Minimisation of the Dirichlet Energy amongst a class of constrained maps

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Elasticity and Energy Functionals

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where W is the **stored-energy function** of the material.

A common form for W is

$$W(M) = |M|^2 + h(\det M) \qquad M \in \mathbb{R}^{2\times 2}$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ satisfies certain properties that prevents the **interpenetration of matter** ($\det \nabla v \leq 0$).

Simplifcation via Constraint

If we restrict to maps with prescribed Jacobians

$$\det \nabla v = f \qquad \text{a.e.}$$

then we have

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The case of $f \equiv 1$ is of particular interest as it corresponds to **incompressible** deformations (i.e: mass conserving).

Harmonic Maps and Mass Conservation

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Incompressible Harmonic Maps

Let $u \in H^1(B; \mathbb{R}^2)$ be harmonic. If $\det \nabla u = 1$ then u is **affine**.

Hence, non-affine maps do not minimise the Dirichlet energy amongst incompressible maps.

Excess Functionals

For a given $u \in H^1$, we wish to minimise $\mathbb{D}$ amongst

$$\mathcal{A}_u = \{v \in H^1_u(B;\mathbb{R}^2) : \det \nabla v = \det \nabla u \quad \text{a.e.} \}$$

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Motivated by the work of J.Bevan, we define the **excess functional** for a given **pressure function** $\mathsf{p} \in \mathrm{BMO}(B)$ as

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \int_B |\nabla \varphi|^2 + \mathsf{p} \det \nabla \varphi \, \mathrm{d}x \qquad \varphi \in H^1_0(B;\mathbb{R}^2)$$

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Decomposition via Excess

Take $\mathrm{p} \in \mathrm{BMO}(B)$ such that u solves the Euler-Lagrange equation for $\mathbb{E}_{\mathrm{p}}$. Then

$$\mathbb{D}(v) = \mathbb{D}(u) + \mathbb{E}_{\mathrm{p}}(v - u) \qquad \forall v \in \mathcal{A}_u$$

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The case of *small* pressure is trivial:

Small Pressure

If $\mathrm{p} \in L^\infty$ satisfies $\|\mathrm{p} - \overline{\mathrm{p}}\|_\infty \leq 2$ then $\mathbb{E}_{\mathrm{p}} \geq 0$.

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Polyconvexity

We immediately observe that the integrand of $\mathbb{E}_\mathsf{p}$ is **polyconvex**.

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Convexity

Consider a function $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$. We say W is **convex** if

$$W(\lambda M + (1-\lambda)N) \leq \lambda W(M) + (1-\lambda)W(N) \qquad \lambda \in [0,1]$$

for all $M, N \in \mathbb{R}^{2\times 2}$. We say W is **polyconvex** if

$$W(M) = w(M, \det M)$$

with w convex on $\mathbb{R}^{2\times 2} \times \mathbb{R}$. We say W is **quasiconvex** if

$$⨍_B W(M + \nabla\varphi)\,\mathrm{d}x \geq W(M) \qquad \forall \varphi \in W_0^{1,\infty}(B;\mathbb{R}^2)$$

for all $M \in \mathbb{R}^{2\times 2}$.

The Direct Method

The **direct method** is the classical method for establishing existence of minimisers.

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There are many variants of the direct method, including versions designed for polyconvex integrands, e.g:

Direct Method

Let $\mathcal{A}$ be non-empty and $I : \mathcal{A} \to \mathbb{R}$ be given an integral functional with integrand W depending only on the gradient. If

- W is polyconvex and
- W satisfies the coercivity condition:

$$W(M) \geq c_0 \left(|M|^2 + |\det M|^{\frac{3}{2}} \right) - c_1 \qquad \forall M \in \mathbb{R}^{2\times 2}$$

for some $c_0 > 0$ and $c_1 \in \mathbb{R}$, then I has a minimiser in $\mathcal{A}$.

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For an excess functional with *large* pressure, this is non-trivial.

This motivates the exploration of novel techniques to establish bounds for functionals of a polyconvex integrand, in particular $\mathbb{E}_p$.

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Fourier Series

One of the main techniques implemented is decomposing quantities using Fourier modes.

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Fourier Series

Let $u \in L^2(B; \mathbb{R}^2)$. Then u has a Fourier series representation

$$u = \sum_{j \geq 0} U_j(r) e_r(j\theta)$$

with **mode matrices** given by

$$U_j(r) = (2 - \delta_j) \fint_0^{2\pi} u \otimes e_r(j\theta) \, \mathrm{d}\theta$$

Note that $\det U_0 = 0$ since the second column of U_0 vanishes.

Sufficient Conditions

Assuming radially symmetric p, we decompose $\mathbb{E}_{\mathsf{p}}$ as

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \pi \sum_{j \geq 0} \int_0^1 (1+\delta_j) r \left|\Phi_j'\right|^2 + \frac{j^2}{r} \left|\Phi_j\right|^2 - j \mathsf{p}'(r) \det \Phi_j \, \mathrm{d}r$$

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Applying a weighted Poincaré inequality and Hadamard's inequality pointwise, we get a sharp estimate:

$$\mathbb{E}_{\mathsf{p}}(\varphi) \geq \pi \sum_{j \geq 0} \int_0^1 \left((1+\delta_j) j_0^2 r + \frac{j^2}{r} - \frac{j \left|\mathsf{p}'(r)\right|}{2} \right) \left|\Phi_j\right|^2 \, \mathrm{d}r$$

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where j_0 is the first postive zero of the Bessel function J_0.

Sufficient Condition

$$\left|\mathsf{p}'(r)\right| \leq \begin{cases} 2j_0^2 r + \frac{2}{r} & r \leq \frac{1}{j_0} \\ 4j_0 & r \geq \frac{1}{j_0} \end{cases} \implies \mathbb{E}_\mathsf{p} \geq 0$$

Example - Radially Affine Pressure

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This is an improvement on the result using pointwise estimates:

$$|\lambda| \leq 3$$

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$$\mathbb{E}_{\mathsf{p}}(\varphi) = 2\,|B| + \int_{B^+} \mathsf{p}\,\mathrm{d}x - \int_{B^-} \mathsf{p}\,\mathrm{d}x$$

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Necessary Condition

$$\mathbb{E}_{\mathsf{p}} \geq 0 \quad \Longrightarrow \quad \left| \fint_{B^+} \mathsf{p}\,\mathrm{d}x - \fint_{B^-} \mathsf{p}\,\mathrm{d}x \right| \leq 4$$

The other side of the inequality is gained by substituting $\varphi(x) \mapsto \varphi(I^- x)$ where I^- is a reflection matrix.

Construction of Solutions

To make this condition concrete, we need to construct a solution to the PDI

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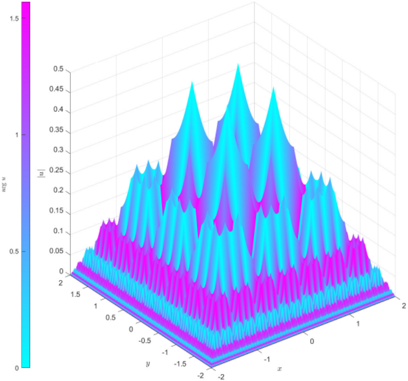
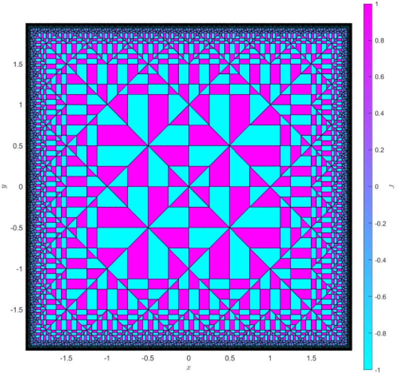
If we use a partial covering we have to alter the necessary condition accordingly.

Construction of Solutions





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For non-negativity of $\mathbb{E}_{\mathsf{p}_\lambda}$ we have that

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There is plenty of room to improve the necessary condition by trying different tilings.

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If fact, we can also derive a constrained optimisation problem from an excess functional.

Existence of Candidate Minimiser

Let $\mathrm{p} : B \to \mathbb{R}$ be radially symmetric and *sufficiently regular*. Then, *dependent on the choice of boundary condition*, there exists a unique $u \in H^1(B; \mathbb{R}^2)$ such that u satisfies the Euler-Lagrange equation for $\mathbb{E}_{\mathrm{p}}$.

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Here, any pressure function of the form

$$\mathrm{p}(r) = \mathrm{p}_* \log r + a(r) \qquad a \in C^\omega([0,1]; \mathbb{R})$$

is sufficiently regular.

Boundary Condition and Conformality

The derived candidate minimiser u has mode matrix U_j taking the form

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If $U_j(1) \notin \mathrm{CO}_-(2)$ then α_j must satisfy

$$r^2 \alpha_j'' + r\alpha_j' + \left(\frac{j}{2} r \mathrm{p}' - j^2\right) \alpha_j = 0 \qquad \alpha_j(0) = 0 \quad \alpha_j(1) = 1$$

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If $U_j(1) \notin \mathrm{CO}_+(2)$ then β_j must satisfy

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The value of p_* plays a significant role in existence.

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Summary

To summarise, we have shown that is it possible to construct maps u such that we can minimise $\mathbb{D}$ amongst a class of Jacobian constrained maps $\mathcal{A}_u$.