Wirtz Pumps and Infinite Dimensions

An Introduction to the Calculus of Variations

Elliott Farrall

University of Surrey

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Inspiration & YouTube Stardom

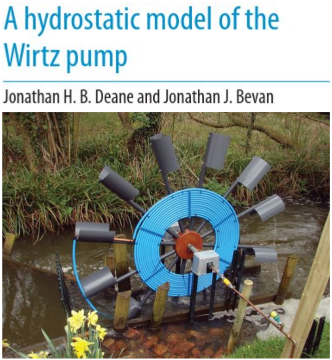

Figure: Wirtz pump in J.D's garden

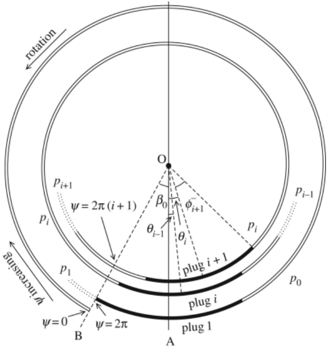

Figure: Diagram taken from paper

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What is the optimal shape of the spiral to create the most pressure?

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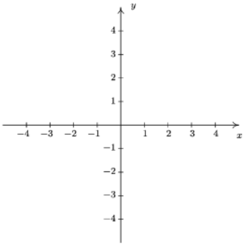

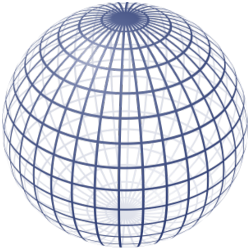

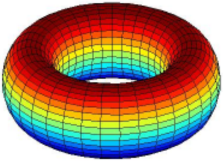

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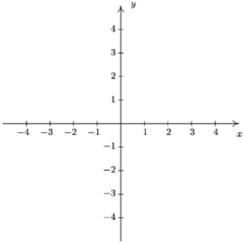

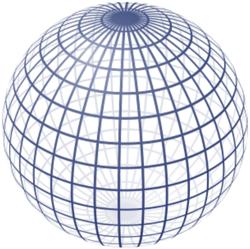

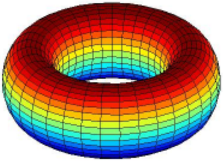

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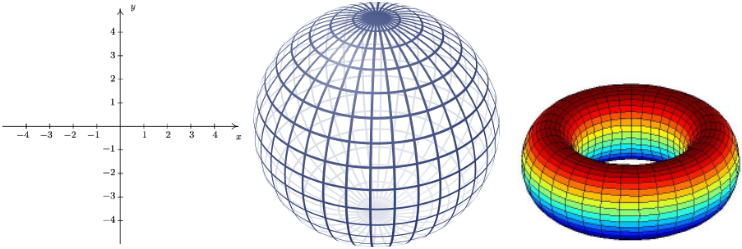

A **function** $f : D \to \mathbb{R}$ can be optimised (locally) by finding zeros of the derivative.

Global optimisation is done by comparing local optima along with boundary values.

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All these spaces come with a **metric**, i.e: we can measure distance between any two elements of them.

Weak Differentiability

The **weak derivative** v of a function u is defined through IBP against test functions $\phi \in C_c^\infty(D)$

$$\int_D u\phi' \,\mathrm{d}x = -\int_D v\phi \,\mathrm{d}x$$

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Note that this definition only requires u to be defined **almost everywhere**.

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For example, $H^1 = W^{1,2}$, the space of functions with $u, u' \in L^2$.

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However, it has numerous applications in various physical sciences, aswell as being a beautiful area of mathematics in its own right.

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If we specify some initial conditions, we can solve this equation to find the path of the particle.

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Again, the minimising curve can found by solving a differential equation:

$$\left(1 + y'(x)^2\right) y(x) = k^2$$

where k is a constant that is determined by the start/end point.

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It minimises the Dirichlet energy functional:

$$E[\mathbf{u}] = \int_D \frac{1}{2} |\nabla \mathbf{u}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}$$

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The augmented functional for such a problem is not even convex.

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