Optimisation of a 1D Family of Polyconvex Functionals

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- Elastic materials take configurations that minimise an energy functional.
- When an elastic incompressible material is stretched, we can sometimes split the energy into the energy of a candidate minimiser and an excess functional.
- We seek to find conditions under which we can find a global minimiser of such functionals.

Defining the Functional

Let B be the unit ball in $\mathbb{R}^2$. We define functionals I_f

$$I_f(u) = \int_B |\nabla u|_{\mathrm{F}}^2 + f(x) \det \nabla u \, \mathrm{d}x \quad u \in W_0^{1,2}(B; \mathbb{R}^2)$$

for $f \in \mathrm{Lip}(B; \mathbb{R})$.

- When $f = 0$ we recover the Dirichlet energy functional
$$I_0(u) = \int_B |\nabla u|_{\mathrm{F}}^2 \, \mathrm{d}x$$
- If I_f is negative at any point, it is unbounded below.
- Since $I_f(0) = 0$, it is sufficient to show that $I_f \geq 0$ to demonstrate the existence of a global minimiser.
- Take $I_\lambda := I_f$ with $f(x) = \lambda |x|_2$.

Additional Materials

If you would like see some animated plots or the references for this poster, please scan the QR code.

Sufficient Conditions

Use Hadamard's inequality [3] pointwise, to get

$$I_\lambda(u) \geq \int_B \left(1 - \frac{\lambda}{2} |x|_2\right) |\nabla u(x)|_{\mathrm{F}}^2 \, \mathrm{d}x$$

Hence, $\lambda \leq 2$ is sufficient for $I_\lambda \geq 0$.

Alternatively, decompose [1] $u \in W_0^{1,2}(B; \mathbb{R}^2)$ as

$$u(x) = \sum_{j \geq 0} u^{(j)}(x) = \sum_{j \geq 0} \mathbf{A}_j(r) \cos(j\theta) + \mathbf{B}_j(r) \sin(j\theta)$$

Use a weighted Poincaré inequality for the Fourier modes:

$$\int_B \left| u^{(j)} \right|_2^2 \mathrm{d}x \leq \frac{1}{J_0^2} \int_B \left| u_{,r}^{(j)} \right|_2^2 \mathrm{d}x \qquad j \geq 0$$

where J_0 is the first zero of a Bessel function. Then

$$I_\lambda(u) \geq \sum_{j \geq 0} \int_B \mathbf{U}_j^\intercal M_\lambda \mathbf{U}_j \, \mathrm{d}x \qquad \mathbf{U}_j = \begin{pmatrix} \left| u^{(j)} \right|_2 \\ \left| u_{,\tau}^{(j)} \right|_2 \end{pmatrix}$$

Thus, $M_\lambda \geq 0$ is sufficient for $I_\lambda \geq 0$.
Hence, $\lambda \leq 4J_0 \approx 9.619$ is sufficient.

Necessary Conditions

We consider a specific $u^* \in W_0^{1,2}(B; \mathbb{R}^2)$ satisfying:

- $u^* = 0$ on some region $K \subset B$.
- $\nabla u^* \in \mathrm{O}(2)$ on some squares $Q_i \subset B$.

Then a necessary condition is

$$\lambda \leq \frac{2(\pi - |K|)}{|\mathcal{I}_0|}$$

where $\mathcal{I}_0$ is given by an integral of $\det \nabla u^*(x) \, |x|_2$ over the squares Q_i.

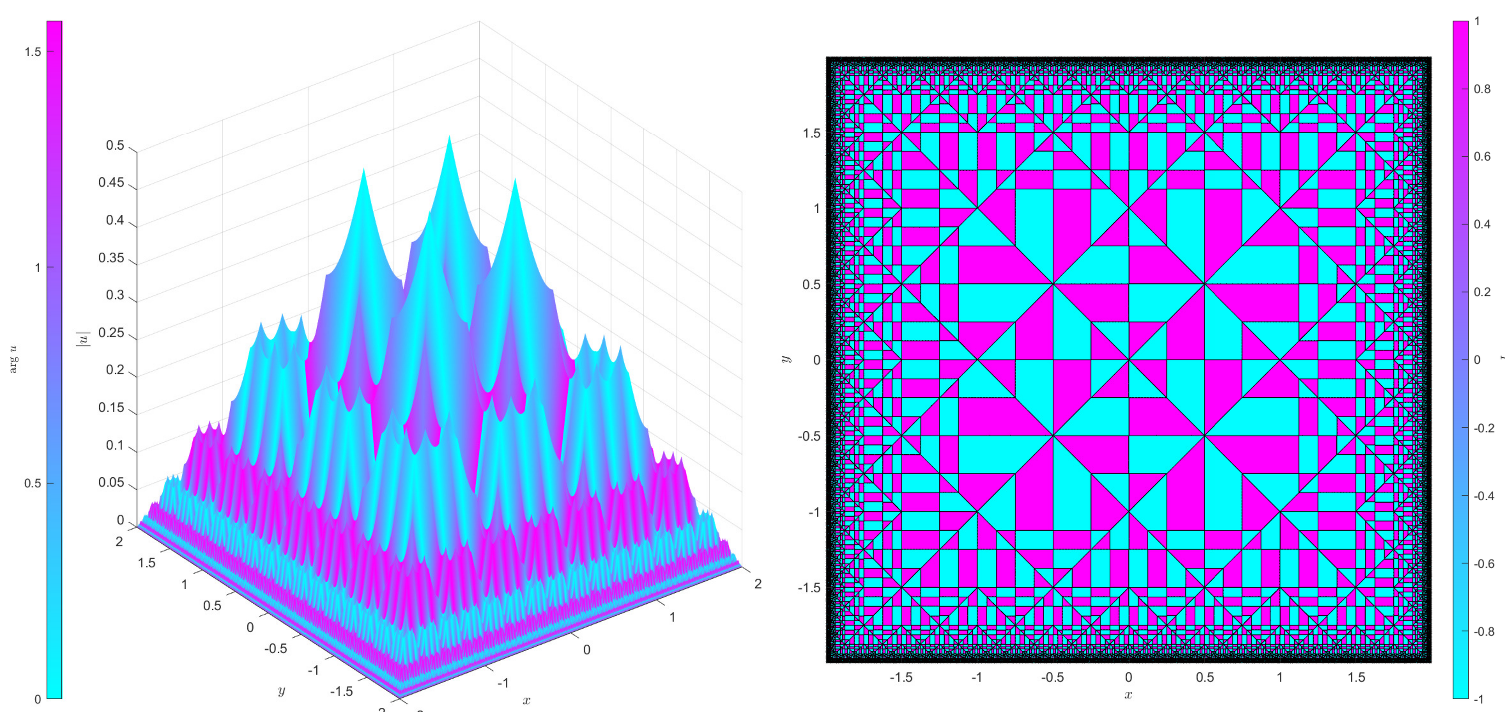
Constructing Solutions to $\nabla u \in \mathrm{O}(2)$

An example of the **problem of potential wells** [2, 4, 5].

Theorem (Dacorogna and Marcellini) [3]: There exists a dense set of solutions in $W^{1,\infty}(Q; \mathbb{R}^2)$

Theorem (Liouville) [4]: All solutions are piecewise affine.

We adapt an explicit 3D [2] solution to 2D.



Take two squares with centres and widths:

$$c_1 = \begin{pmatrix} +0.1 \\ +0.3 \end{pmatrix} \quad w_1 = 1.0 \qquad c_2 = \begin{pmatrix} -0.3 \\ -0.5 \end{pmatrix} \quad w_1 = 0.6$$

We obtain $|\mathcal{I}_0| = 1.83384 \times 10^{-2} \pm 2 \times 7.62939 \times 10^{-6}$.
Hence we require $\lambda \leq 148.446$.

Conclusions

- We have improved on the bound for λ that is sufficient for $I_\lambda \geq 0$ given by using Hadamard's inequality pointwise.
- We have found a bound for λ that is necessary for $I_\lambda \geq 0$ by constructing a solution to a previously studied PDI.
- This bound may be further improved by finding a different solution to $\nabla u \in \mathrm{O}(2)$ or using an alternative configuration of squares Q_i.