Conditions for the Non-Negativity of Excess Functionals in Constrained Variational Problems

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PhD Thesis submitted to the University of Surrey

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Scientific abstract

In this thesis, we will study a family of functionals, referred to as excess functionals, introduced by J Bevan, M Kružìk and J Valdman [10] to derive so-called mean Hadamard inequalities, a functional generalisation of the classical (pointwise) Hadamard inequality. In turn, these inequalities may be used to construct examples of Jacobian constrained variational problems and minimisation problems for functionals of a polyconvex integrand, that have explicit solutions. This is of particular relevance in the field of elasticity [7], where these functionals describe the stored energy associated with the deformation of a material and Jacobian constraints are a form of incompressibility condition.

The focus of the thesis will be primarily on developing explicit and constructive methods for deriving examples and counter-examples of mean Hadamard inequalities. We will also describe how to use these mean Hadamard inequalities to form Jacobian constrained classes in which the Dirichlet energy has a global minimiser. In the cases where we can not show the existence of a minimiser, we will show that the techniques used to bound the excess functional can also be used to bound the Dirichlet energy on the constrained class. The results will mainly be for the case of a ball shaped domain in two dimensions but can be generalised using conformal mappings.

Keywords: Calculus of Variations, constrained variational problems, elasticity, polyconvexity, mean Hadamard inequality, partial differential inclusions.

AMS Classification Codes: 49J40, 49K10, 49K15, 49K20, 49K24.

Acknowledgements

I would like to express my sincere gratitude to my supervisors, Dr. J. Bevan and Prof. J. Deane, for their expert guidance and steadfast support throughout this journey. Their insights and mentorship have been essential to the completion of this work. I am also grateful to my confirmation examiners, Dr. J. Grant and Dr. B. Cheng, for their thoughtful feedback and constructive suggestions, which have greatly enhanced this thesis. My thanks go to my colleagues and peers for their valuable discussions and collaborative spirit, which enriched this research and made the process more enjoyable.

To my wife, Chloe, your patience and unwavering belief in me have been a constant source of strength. I am truly grateful for your support throughout this academic journey. I also wish to thank my friends for their understanding, encouragement, and the much-needed breaks that made this time more manageable and memorable. Finally, I extend my deepest appreciation to my family for their ongoing love, support, and belief in my potential. Their sacrifices and encouragement have been the foundation of everything I have achieved.

Declaration

This thesis and the work to which it refers are the results of my own efforts. Any ideas, data, images or text resulting from the work of others (whether published or unpublished, and including any content generated by a deep learning/artificial intelligence tool) are fully identified as such within the work and attributed to their originator in the text, bibliography or in footnotes. This thesis has not been submitted in whole or in part for any other academic degree or professional qualification. I agree that the University has the right to submit my work to the plagiarism detection service TurnitinUK for originality checks. Whether or not drafts have been so assessed, the University reserves the right to require an electronic version of the final document (as submitted) for assessment as above.

For my father Andy

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1 Introduction

We start by reviewing Hadamard's pointwise inequality and generalising it to a functional inequality. This will motivate the definition of the excess functional, which will be the primary focus of this thesis. We will then further motivate the study of this functional as a tool for investigating certain forms of constrained minimisation problems in the Calculus of Variations and, in particular, non-linear elasticity. We will also review the various notions of convexity and the role they play in establishing the existence of minimisers in variational problems. The excess functional will serve as an example of a functional that is not covered by existing results that use the Direct Method, for example, but for which we can show the existence of minimisers through other methods.

1.1. Mean Hadamard Inequalities

Hadamard's inequality [23, 37] is a two-sided inequality that bounds the determinant of a matrix in terms of its Frobenius norm,

$$|\det M| \leqslant \frac{1}{2}|M|^2 \qquad \forall M \in \mathbb{R}^{2\times 2}. \tag{1.1}$$

This inequality is sharp, with equality attained on one side by taking $M \in \mathrm{CO}^+(2)$, the space of conformal matrices, and on the other by taking $M \in \mathrm{CO}^-(2)$, the space of anti-conformal matrices, given by

$$\mathrm{CO}^{\pm}(2) := \left\{X \in \mathbb{R}^{2\times 2} : \operatorname{cof} X = \pm X\right\}.$$

Inequality (1.1 generalises to n dimensions, with the constant $\frac{1}{2}$ on the right-hand side being replaced with $n^{-\frac{n}{2}}$ but we will stick to $n = 2$ dimensions. This inequality can be shown to be a consequence of the arithmetic-geometric mean inequality applied to the squares of the singular values [30, 39].

If we replace the constant matrix M with the gradient of a map $\varphi \in H^1(B;\mathbb{R}^2)$ and integrate over the unit ball $B \subset \mathbb{R}^2$, this inequality can be written as

$$\int_B \frac{1}{2}|\nabla\varphi(x)|^2 \pm \det\nabla\varphi(x)\,\mathrm{d}x \geqslant 0 \qquad \forall\varphi \in H^1(B;\mathbb{R}^2).$$

We could of course pick a different domain in $\mathbb{R}^2$, but we will focus mainly on the case of a ball for simplicity. A natural question would then be whether the coefficient of $\det\nabla\varphi$ is still optimal in this new, mean inequality. More precisely, we ask for which functions $\mathsf{p} : B \to \mathbb{R}$ do we have

$$\mathbb{E}_{\mathsf{p}}(\varphi) := \int_B \frac{1}{2}|\nabla\varphi(x)|^2 + \mathsf{p}(x)\det\nabla\varphi(x)\,\mathrm{d}x \geqslant 0 \qquad \forall\varphi \in H^1_0(B;\mathbb{R}^2). \tag{1.2}$$

We shall refer to the functional $\mathbb{E}_{\mathsf{p}}$ as the excess functional and the parameter function p as the pressure function, for reasons that will soon become clear. The justification of the vanishing Dirichlet boundary condition shall also be deferred for now. We will refer to an inequality of the form (1.2) as a *mean Hadamard inequality*. Using (1.1), we immediately find a class of pressure functions for which (1.2) holds.

Corollary 1.1. *Let* $\mathsf{p} \in L^\infty(B)$ *satisfy* $\|\mathsf{p} - \bar{\mathsf{p}}\|_\infty \leqslant 1$. *Then* $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

Proof. We shall exploit the fact that $\varphi \mapsto \det\nabla\varphi$ is a null-Lagrangian [21] and write

$$\begin{aligned}\mathbb{E}_{\mathsf{p}}(\varphi) &= \int_B \frac{1}{2}|\nabla\varphi(x)|^2 + (\mathsf{p}(x) - \bar{\mathsf{p}})\det\nabla\varphi(x)\,\mathrm{d}x + \bar{\mathsf{p}}\int_B \det\nabla\varphi(x)\,\mathrm{d}x\\ &= \int_B \frac{1}{2}|\nabla\varphi(x)|^2 + (\mathsf{p}(x) - \bar{\mathsf{p}})\det\nabla\varphi(x)\,\mathrm{d}x.\end{aligned}$$

If $\|\mathsf{p} - \bar{\mathsf{p}}\|_\infty \leqslant 1$, then the integrand is non-negative for almost every $x \in B$ by (1.1). Hence $\mathbb{E}_{\mathsf{p}} \geqslant 0$. □

Due to a duality result [29, 38] that will be discussed later, we note that $\mathbb{E}_{\mathsf{p}}(\varphi)$ is finite for every $\varphi \in H^1_0(B;\mathbb{R}^2)$ if $\mathsf{p} \in \mathrm{BMO}(B)$. Going forward, we will assume that p has at leat BMO regularity but, in some cases, we will assume additional regularity.

The case of a piecewise constant pressure on a square domain has been discussed in great depth in the literature [10]. There it is shown that there exist pressure functions $\mathsf{p} : [-1,+1]^2 \to \mathbb{R}$ with

$$\|\mathsf{p} - \bar{\mathsf{p}}\|_\infty \leqslant \sqrt{2},$$

such that $\mathbb{E}_{\mathsf{p}} \geqslant 0$. This result can not be obtained by just applying Hadamard's inequality pointwise as we did in the case of Corollary 1.1 since $1 < \sqrt{2}$. Furthermore, this shows that there exist non-trivial mean Hadamard inequalities which can be proven using methods that go beyond the application of Hadamard's pointwise inequality. There are also necessary condition results for piecewise constant pressure [10] and we will make use of the techniques used to prove these results for other families of pressure functions.

There has also been some discussion [9] of the case of a radially symmetric, logarithmic pressure function

$$\mathsf{p}(x) = \mathsf{p}_* \log(|x|) \qquad c > 0.$$

This is an example of a lower regularity pressure function that fails to be bounded but is still in BMO. Trivially, these pressure functions do not satisfy the pre-requisite condition for Corollary 1.1. However, we will show that these pressure functions do give rise to a mean Hadamard inequality, provided the coefficient p_* is picked sufficiently small (see sections 2.4.2. and 3.3.2.).

Proposition 1.2. *If* $|\mathsf{p}_*| \leqslant 1$*, then* $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

1.2. Constrained Variational Problems and Excess Functionals

It turns out that the functional $\mathbb{E}_{\mathsf{p}}$ arises naturally when considering a certain form of constrained variational problem. More specifically, we consider the problem of minimising the Dirichlet energy

$$\mathbb{D}(v) = \int_B \frac{1}{2} |\nabla v(x)|^2 \, \mathrm{d}x \qquad v \in H^1(B; \mathbb{R}^2)$$

subject to a Jacobian constraint

$$\det \nabla v(x) = h(x) \qquad \text{a.e. } x \in B, \tag{1.3}$$

and Dirichlet boundary condition

$$v(x) = g(x) \qquad \forall x \in \partial B.$$

The notion of minimising energy while controlling the Jacobian is a common occurrence in elasticity [7]. Here v would represent the deformation of a disc-shaped material relative to a coordinate system. Taking $h \equiv 1$, the condition (1.3) would then become an incompressibility constraint.

We will reformulate the variational problem by introducing a map $u \in H^1(B; \mathbb{R}^2)$ such that

$$\begin{aligned} \det \nabla u(x) &= h(x) \qquad \text{a.e. } x \in B, \\ u(x) &= g(x) \qquad \forall x \in \partial B. \end{aligned}$$

There has been some discussion [19] of the existence of such a u for the case of $g = \mathrm{id}$. We will not concern ourselves with the existence of u but, instead, let u be the defining parameter for this class of constrained variational problems, instead of (g, h). We then define the space of admissible maps $\mathcal{A}_u$ by

$$\mathcal{A}_u := \left\{ v \in H^1_u(B; \mathbb{R}^2) : \det \nabla v(x) = \det \nabla u(x) \ \text{ a.e. } x \in B \right\},$$

and so the constrained variational problem is simply

$$\min_{v \in \mathcal{A}_u} \mathbb{D}(v).$$

We note that the class $\mathcal{A}_u$ is non-empty as it contains u, and is in fact an equivalence class in $H^1(B;\mathbb{R}^2)$ with representative u for the equivalence relation

$$u \sim v \qquad \Longleftrightarrow \qquad \begin{cases} \det \nabla u(x) = \det \nabla v(x) & \text{a.e. } x \in B, \\ u(x) = v(x) & \forall x \in \partial B. \end{cases}$$

As an immediate corollary, we observe that $\mathcal{A}_u = \mathcal{A}_v$ for all $v \in \mathcal{A}_u$. The non-triviality of $\mathcal{A}_u$ (as in $|\mathcal{A}_u| > 1$) has been shown for some examples of u [9] and we will not be exploring this for more general classes of u. To see how the excess functional and pressure functions fit in to this picture, observe the following result.

Theorem 1.3. *Let* $\mathsf{p} : B \to \mathbb{R}$ *be such that* u *solves the weak Euler-Lagrange equation for* $\mathbb{E}_{\mathsf{p}}$*. Then*

$$\mathbb{D}(v) = \mathbb{D}(u) + \mathbb{E}_{\mathsf{p}}(v - u) \qquad \forall v \in \mathcal{A}_u.$$

Proof. The weak Euler-Lagrange equation [21] for $\mathbb{E}_{\mathsf{p}}$ is given by

$$\int_B (\nabla u + \mathsf{p}\,\mathrm{cof}(\nabla u)) \cdot \nabla\varphi \,\mathrm{d}x = 0 \qquad \forall \varphi \in \mathcal{T}, \tag{1.4}$$

where $\mathcal{T}$ is a space of test functions. Now suppose that $v \in \mathcal{A}_u$ and let $\varphi = v - u$. We observe that

$$\begin{aligned} \mathbb{D}(v) = \mathbb{D}(u+\varphi) &= \mathbb{D}(u) + \int_B \nabla u \cdot \nabla\varphi \,\mathrm{d}x + \mathbb{D}(\varphi) \\ &= \mathbb{D}(u) - \int_B \mathsf{p}\,\mathrm{cof}\,\nabla u \cdot \nabla\varphi \,\mathrm{d}x + \mathbb{D}(\varphi) \end{aligned}$$

using (1.4) and approximating φ by smooth functions where necessary. Next, we use the constraint to calculate

$$0 = \det(\nabla u + \nabla\varphi) - \det(\nabla u) = \det\nabla\varphi + \mathrm{cof}(\nabla u)\cdot\nabla\varphi \quad \Rightarrow \quad \det\nabla\varphi = -\,\mathrm{cof}(\nabla u)\cdot\nabla\varphi.$$

Hence, we find that

$$\mathbb{D}(v) = \mathbb{D}(u) + \int_B \mathsf{p}\det\nabla\varphi\,\mathrm{d}x + \mathbb{D}(\varphi) = \mathbb{D}(u) + \mathbb{E}_{\mathsf{p}}(\varphi).$$

□

As an immediate corollary, we obtain a sufficient condition for u to minimise $\mathbb{D}$ in $\mathcal{A}_u$.

Corollary 1.4. *Let* $\mathsf{p} : B \to \mathbb{R}$ *be such that* u *solves the weak Euler-Lagrange equation for* $\mathbb{E}_{\mathsf{p}} \geqslant 0$. *Then*

$$\mathbb{D}(u) = \min_{v \in \mathcal{A}_u} \mathbb{D}(v).$$

In other words, u *is a global minimiser of* $\mathbb{D}$ *in* $\mathcal{A}_u$.

Proof. If $\mathbb{E}_{\mathsf{p}} \geqslant 0$, then

$$\mathbb{D}(v) = \mathbb{D}(u) + \mathbb{E}_{\mathsf{p}}(v - u) \geqslant \mathbb{D}(u) \qquad \forall v \in \mathcal{A}_u.$$

□

Assuming a sufficiently regular pressure function, say $\mathsf{p} \in W^{1,1}(B; \mathbb{R}^2)$, we can write down a strong form of the Euler-Lagrange equation for $\mathbb{E}_{\mathsf{p}}$, given by

$$\Delta u + \operatorname{cof}(\nabla u)\nabla \mathsf{p} = 0 \qquad u \in H^1(B; \mathbb{R}^2). \tag{1.5}$$

This is a second order, linear, elliptic, coupled system of PDEs in u. We will not explore the general theory for PDEs taking this form but will explore methods for decoupling and constructing explicit solutions for u, under some assumptions on p.

Proposition 1.5. *Let* $\mathsf{p} : B \to \mathbb{R}$ *be radially symmetric and satisfy*

$$\mathsf{p}(r) = \mathsf{p}_* \log(r) + \sum_{k \geqslant 0} \mathsf{p}_k r^k, \qquad \forall r \in (0, 1].$$

Then, dependent on the choice of boundary conditions, there exists a solution u *to the Euler-Lagrange equation (1.5).*

For details of the requirements for the boundary conditions, see Proposition 2.12.

Before moving on, we will briefly discuss the non-triviality of the Jacobian constraint (1.3). A naive approach to finding a solution to the variational problem would be to simply minimise the Dirichlet energy subject to the boundary condition, but ignoring the Jacobian constraint, and hope that the obtained minimiser happens to satisfy the constraint. In general, this will not hold, and we will prove this for the case of $\det \nabla u = 1$. Recall that minimisers of the Dirichlet energy, in the absence of any constraint, are harmonic maps [21].

Proposition 1.6. *Let* $u \in H^1(B; \mathbb{R}^2)$ *be harmonic and satisfy*

$$\det \nabla u(x) = 1 \qquad a.e.\ x \in B.$$

Then

$$u(x) = Mx + c,$$

for constant $M \in \mathrm{SL}(2)$ *and* $c \in \mathbb{R}^2$.

Proof. We start by identifying $\mathbb{R}^2$ with $\mathbb{C}$ and map u into the complex plane,

$$u(x) = (u_1(x_1, x_2), u_2(x_1, x_2))^{\mathsf{T}} \quad \mapsto \quad u(z) = u_1(x_1, x_2) + u_2(x_1, x_2)i \qquad z = x_1 + x_2 i.$$

Then we can can write

$$u = f + \bar{g},$$

for holomorphic functions f, g [24, 1. Introduction]. More explicitly, we have

$$f = \frac{u_1 + \hat{u}_1}{2} + \frac{u_2 + \hat{u}_2}{2} i, \qquad g = \frac{u_1 - \hat{u}_1}{2} - \frac{u_2 - \hat{u}_2}{2} i,$$

with the hat denoting harmonic conjugate [13], that is, $u_j + \hat{u}_j i$ satisfy the Cauchy-Riemann equations [13]. Then we calculate

$$1 = \det \nabla u = |u_z|^2 - |u_{\bar{z}}|^2 = \left|f'\right|^2 - \left|g'\right|^2.$$

Hence we have that

$$\left|f'\right|^2 = 1 + \left|g'\right|^2.$$

We shall now denote $F = (f')^2$ and $G = (g')^2$, which are both holomorphic, and satisfy

$$|F| = |G| + 1. \tag{1.6}$$

We can also think of F and G as mappings of $\mathbb{R}^2$, which allows us to calculate, using the fact F and G are holomorphic,

$$\left|F'\right| = |\nabla |F|| = |\nabla |G|| = \left|G'\right|,$$

and so $F' = \lambda G'$ for some $\lambda \in S^1$. Furthermore, we must have λ constant. To show this we consider several possible cases.

- First, we suppose there exists a disk in B for which $G' \neq 0$. On this disk, we have that

  $$\lambda = \cos(\theta(x, y)) + i\sin(\theta(x, y)) = \frac{F'}{G'},$$

  is holomorphic. Hence, λ satisfies the Cauchy-Riemann equations, which we can write as

  $$\begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ +\cos(\theta) & -\sin(\theta) \end{pmatrix} \nabla\theta = 0.$$

  Since the coefficient matrix is a rotation matrix, we observe that $\nabla\theta = 0$ so θ, and hence λ, is constant. Thus we have that $F = \lambda G + \mu$ for a constant $\mu \neq 0$, as $|F| \neq |G|$. Substituting this back in to 1.6, we find that

  $$|F| = \left|\frac{F - \mu}{\lambda}\right| + 1 \Rightarrow 1 + |F - \mu| = |F|.$$

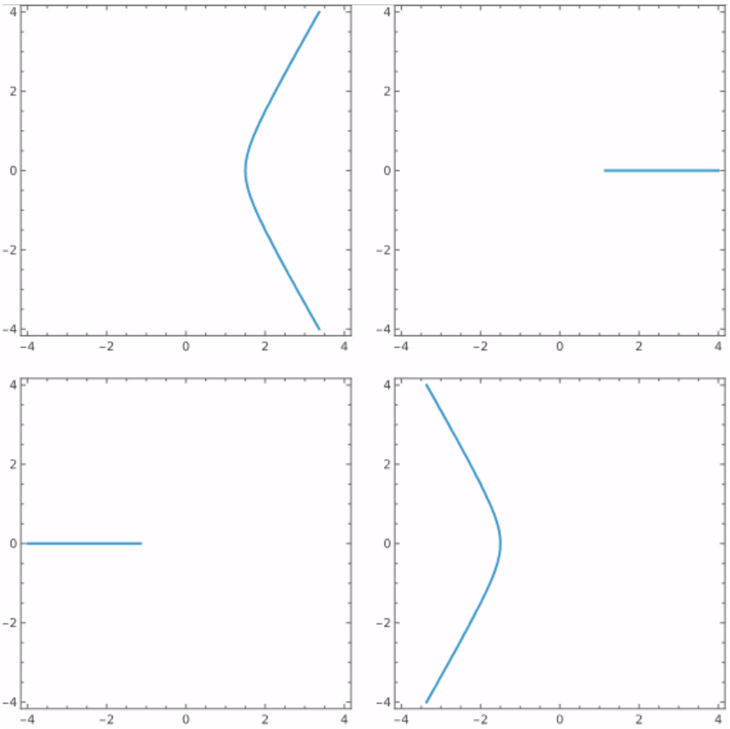

Figure 1.1: Plot of the locus (1.7) for $\mu = -2, -1, 1, 2$ from left to right and top to bottom.

However, the locus of

$$z \mapsto 1 + |z - \mu| = |z|\,, \tag{1.7}$$

is, at most, one-dimensional. In particular, it is one-dimensional for $|\mu| \geqslant 1$ and empty otherwise (see Figure 1.1). This means that we have an open map F with a closed image, so it must be constant, by the open mapping theorem. Similarly, we must have that G is constant.

- If there exists a disk in B for which $F' \neq 0$, we can proceed as we did before but with $\lambda \in S^1$ replaced by $\lambda^{-1} \in S^1$.
- If both F' and G' vanish almost everywhere in B, then F and G are constant almost everywhere in B. By continuity they are both constant on B.

Hence f, g are both affine and so u is affine. Since $\det \nabla u = 1$, we must have that

$$u(x) = Mx + c,$$

for $M \in \mathrm{SL}(2)$ and $c \in \mathbb{R}^2$ both constant. □

Thus, any map u that is not of the form

$$u(x) = Mx + c \qquad M \in \mathrm{SL}(2) \quad c \in \mathbb{R}^2,$$

and minimises the Dirichlet energy in $H^1(B;\mathbb{R}^2)$ subject to Dirichlet boundary conditions will not satisfy the constraint $\det \nabla u = 1$.

1.3. Convexity and the Direct Method

Establishing lower bounds for the functional $\mathbb{E}_{\mathrm{p}}$ is also an interesting problem in the Calculus of Variations due to the lack of convexity in the integrand. To understand this significance, we must first review the notion of convexity [17] and its role in the Direct Method.

Definition 1.7. *A function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is:*

- *convex, if it satisfies*

$$f(\lambda X + (1-\lambda)Y) \leqslant \lambda f(X) + (1-\lambda) f(Y) \qquad \forall \lambda \in [0,1],$$

  for each $X, Y \in \mathbb{R}^{2\times 2}$.

- *polyconvex, if $f(X) = g(X, \det X)$ for some convex function $g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$.*

- *quasiconvex, if it satisfies*

$$f(X) \leqslant \fint_B f(X + \nabla \varphi(x))\,\mathrm{d}x \qquad \forall \varphi \in W_0^{1,\infty}(B;\mathbb{R}^2).$$

- *rank-one convex, if it satisfies*

$$f(\lambda X + (1-\lambda)Y) \leqslant \lambda f(X) + (1-\lambda) f(Y) \qquad \forall \lambda \in [0,1],$$

  for each $X, Y \in \mathbb{R}^{2\times 2}$ such that $\mathrm{rk}\,(X - Y) = 1$.

These definitions obey the following sequence of implications [17]

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank-one convex}.$$

We note that the integrand of $\mathbb{E}_{\mathrm{p}}$ is convex iff $|\mathrm{p}| \leqslant 1$ but is only polyconvex otherwise. Since $\mathbb{E}_{\mathrm{p}}(0) = 0$ and $\mathbb{E}_{\mathrm{p}}$ is homogeneous of degree two, the bound $\mathbb{E}_{\mathrm{p}} \geqslant 0$ is equivalent to the existence of a global minimiser for $\mathbb{E}_{\mathrm{p}}$. In particular, 0 would be the global minimum.

Proposition 1.8. *Let $\mathrm{p} \in \mathrm{BMO}(B)$. Then*

$$\mathbb{E}_{\mathrm{p}} \geqslant 0 \quad \Longleftrightarrow \quad \mathbb{E}_{\mathrm{p}} \text{ has a global minimum.}$$

Proof. We first assume that $\mathbb{E}_{\mathrm{p}} \geqslant 0$. Then since $\mathbb{E}_{\mathrm{p}}(0) = 0$, $\varphi = 0 \in H_0^1(B;\mathbb{R}^2)$ is a global minimiser of $\mathbb{E}_{\mathrm{p}}$ and 0 is the global minimum.

Now assume that $\mathbb{E}_{\mathsf{p}}$ has a global minimum, so it is in particular bounded below. Then, for a contradiction, suppose there exists $\varphi \in H_0^1(B;\mathbb{R}^2)$ such that

$$\kappa := \mathbb{E}_{\mathsf{p}}(\varphi) < 0.$$

Then, for any $m < 0$, there exists $\varphi_m \in H_0^1(B;\mathbb{R}^2)$, given by

$$\varphi_m = \sqrt{\frac{m}{\kappa}}\varphi,$$

such that $\mathbb{E}_{\mathsf{p}}(\varphi_m) = m$. Hence $\mathbb{E}_{\mathsf{p}}$ is unbounded below and we have a contradiction. It then follows that $\mathbb{E}_{\mathsf{p}} \geqslant 0$. □

The notion of convexity often plays a pivotal role in proving the existence of minimisers. In particular, convexity is often used in results that make use of the Direct Method [17,34] to establish the existence of a minimiser. To demonstrate this, consider a more general functional, taking the form

$$I_f(\varphi) = \int_B f(x, \nabla\varphi(x))\,\mathrm{d}x \qquad \varphi \in H_0^1(B;\mathbb{R}^2),$$

where $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is a Carathéodory integrand [34].

Theorem 1.9. *[34, Chapter 2] Let $f(x,\cdot)$ be convex for almost every $x \in B$ and satisfy the coercivity condition*

$$f(x, X) \geqslant \mu\,|X|^2 - C \qquad a.e.\ x \in B, \quad \forall X \in \mathbb{R}^{2\times 2},$$

for constants $\mu, C > 0$. Then there exists a minimiser for I_f in $H_0^1(B;\mathbb{R}^2)$.

Theorem 1.10. *[34, Chapter 5] Let $f(x,\cdot)$ be quasiconvex for almost every $x \in B$ and satisfy the growth condition*

$$-\mu\,|X|^2 \leqslant f(x, X) \leqslant \mu\left(|X|^2 + 1\right) \qquad a.e.\ x \in B, \quad \forall X \in \mathbb{R}^{2\times 2},$$

for constant $\mu > 0$. Then there exists a minimiser for I_f in $H_0^1(B;\mathbb{R}^2)$.

We observe that by weakening the convexity requirement from convexity to quasiconvexity, we require a stronger coercivity/growth condition that may fail to hold. Developing an analogue of these results for polyconvex integrands is an ongoing topic of research in the field of elasticity [7, p. 5-8].

2

Differentiable Pressure Functions

We have already seen that, when p is bounded, we can derive sufficient conditions for $\mathbb{E}_{\mathsf{p}} \geqslant 0$ in terms of $\|\mathsf{p} - \bar{\mathsf{p}}\|_\infty$. The goal of this chapter will be to construct sufficient conditions that use derivatives of p in the case that p is differentiable in some sense. We will achieve this through a variety of techniques, some of which will require the assumption of radial symmetry in the pressure function.

2.1. Fourier Series

Consider a map $\varphi \in L^2(B; \mathbb{R}^2)$ written in polar coordinates (r, θ). Since φ is 2π-periodic in θ, we may decompose it as a Fourier series [15] with coefficients depending on r.

$$\varphi = \frac{1}{2}\Phi_0(r)e_1 + \sum_{j>0} \Phi_j(r)e_r(j\theta), \tag{2.1}$$

where $e_r(\cdot)$ is the radial unit vector, $e_1 = e_r(0)$ is the unit vector pointing in the positive x direction and $\Phi_j : [0, 1] \to \mathbb{R}^{2\times 2}$ are the Fourier coefficient matrices. We will stick to the convention of capitalisation for the mode matrices. Grouping Fourier coefficients together in matrices is somewhat non-standard, but will make many of the calculations performed later much simpler and will allow us to write some results in terms of matrix properties like conformality. The mode matrices can be calculated for a given map using

$$\Phi_j(r) = \frac{1}{\pi}\int_0^{2\pi} \varphi \otimes e_r(j\theta)\,\mathrm{d}\theta,$$

where the product on the right hand side is the standard tensor product which, in this case, coincides with the Kronecker [28] or outer product [31]. We immediately observe that

$$\Phi_0(r)e_2 = 0, \tag{2.2}$$

that is, Φ_0 has only zero entries in its right-most column. Here e_2 denotes the unit vector pointing in the positive y direction. One of the reasons the Fourier decomposition is so useful is the orthogonality [20, p. 3] of the basis functions.

Lemma 2.1. *Let $i, j \in \mathbb{N}_0$. Then*

$$\int_0^{2\pi} e_r(i\theta) \otimes e_r(j\theta)\,\mathrm{d}\theta = \pi\delta_{i,j} I + \pi\delta_i\delta_j I^-,$$

where $I^- := \operatorname{diag}(1, -1)$.

Proof. We start by using product-to-sum formulae [1] for the products of the trigonometric functions.

$$\begin{aligned}
e_r(i\theta) \otimes e_r(j\theta) &= \begin{pmatrix} \cos(i\theta)\cos(j\theta) & \cos(i\theta)\sin(j\theta) \\ \sin(i\theta)\cos(j\theta) & \sin(i\theta)\sin(j\theta) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \cos((i-j)\theta) & -\sin((i-j)\theta) \\ \sin((i-j)\theta) & \cos((i-j)\theta) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos((i+j)\theta) & \sin((i+j)\theta) \\ \sin((i+j)\theta) & -\cos((i+j)\theta) \end{pmatrix} \\
&= \frac{1}{2}R((i-j)\theta)I^+ + \frac{1}{2}R((i+j)\theta)I^-,
\end{aligned}$$

where $I^\pm = \operatorname{diag}(1, \pm 1)$ and $R(\theta)$ denotes the matrix representation of a counter-clockwise rotation of θ radians about the origin [6]. Now, using the fact that

$$\int_0^{2\pi} \cos(k\theta)\,\mathrm{d}\theta = 2\pi\delta_k,$$
$$\int_0^{2\pi} \sin(k\theta)\,\mathrm{d}\theta = 0,$$

for $k \in \mathbb{Z}$, we have

$$\int_0^{2\pi} R(k\theta)\,\mathrm{d}\theta = 2\pi\delta_k I, \qquad k \in \mathbb{Z}.$$

Hence,

$$\int_0^{2\pi} e_r(i\theta) \otimes e_r(j\theta)\,\mathrm{d}\theta = \pi\delta_{i-j} I^+ + \pi\delta_{i+j} I^- = \pi\delta_{i,j} I + \pi\delta_i\delta_j I^-.$$

□

We immediately get a corollary for inner products of the basis functions.

Corollary 2.2. *Let $i, j \in \mathbb{N}_0$. Then*

$$\int_0^{2\pi} e_r(i\theta) \cdot e_r(j\theta)\,\mathrm{d}\theta = 2\pi\delta_{i,j},$$

We can also derive orthogonality results that involve arbitrary coefficient matrices.

Lemma 2.3. *Let $i, j \in \mathbb{N}_0$ and $A, B \in \mathbb{R}^{2\times 2}$. Then*

$$\int_0^{2\pi} (Ae_r(i\theta)) \otimes (Be_r(j\theta))\, \mathrm{d}\theta = \pi\delta_{i,j}AB^\mathsf{T} + \pi\delta_i\delta_j AI^- B^\mathsf{T}.$$

Furthermore, if at least one of A and B has only zero entries in its second column, then

$$\int_0^{2\pi} (Ae_r(i\theta)) \otimes (Be_r(j\theta))\, \mathrm{d}\theta = \pi\left(\delta_{i,j} + \delta_i\delta_j\right) AB^\mathsf{T}.$$

Here A and B are constant but they could depend on any variable other than θ, such as r.

Proof. We first observe that

$$((Ax) \otimes (By))_{kl} = A_{k\mu}x_\mu B_{l\nu}y_\nu = A_{k\mu}(x_\mu y_\nu)B^\mathsf{T}_{\nu l} = \left(A(x\otimes y)B^\mathsf{T}\right)_{kl}, \qquad \forall k, l.$$

Hence, using Lemma 2.1, we have

$$\int_0^{2\pi} (Ae_r(i\theta)) \otimes (Be_r(j\theta))\, \mathrm{d}\theta = A\left(\pi\delta_{i,j}I + \pi\delta_i\delta_j I^-\right) B^\mathsf{T} = \pi\delta_{i,j}AB^\mathsf{T} + \pi\delta_i\delta_j AI^- B^\mathsf{T}.$$

Now suppose at least one of A and B has only zero entries in its second columns and denote their first columns by a, b respectively. We then find that

$$AI^\pm B^\mathsf{T} = a\otimes b = AB^\mathsf{T},$$

and so

$$\int_0^{2\pi} (Ae_r(i\theta)) \otimes (Be_r(j\theta))\, \mathrm{d}\theta = \pi\left(\delta_{i,j} + \delta_i\delta_j\right) AB^\mathsf{T}.$$

□

Again, we can form an analogous result for inner products.

Corollary 2.4. *Let $i, j \in \mathbb{N}_0$ and $A, B \in \mathbb{R}^{2\times 2}$. Then*

$$\int_0^{2\pi} (Ae_r(i\theta)) \cdot (Be_r(j\theta))\, \mathrm{d}\theta = \pi\delta_{i,j}A\cdot B + \pi\delta_i\delta_j A * B,$$

where

$$A * B := (AI^-)\cdot B.$$

Furthermore, if A or B has only zero entries in its second column, then

$$\int_0^{2\pi} (Ae_r(i\theta)) \cdot (Be_r(j\theta))\, \mathrm{d}\theta = \pi\left(\delta_{i,j} + \delta_i\delta_j\right) A\cdot B.$$

Now that we have some orthogonality relations, we can use them to do calculations on a given Fourier series.

Lemma 2.5. *Let $\varphi \in H^1(B;\mathbb{R}^2)$ have Fourier coefficients Φ_j. Then*

$$\nabla\varphi = \frac{1}{2}\Phi_0'(r)e_1 \otimes e_r(\theta) + \sum_{j>0} \Phi_j'(r)e_r(j\theta) \otimes e_r(\theta) + \frac{j}{r}\Phi_j(r)e_\theta(j\theta) \otimes e_\theta(\theta), \tag{2.3}$$

$$\operatorname{cof}\nabla\varphi = \frac{1}{2}\operatorname{cof}\Phi_0'(r)e_2 \otimes e_\theta(\theta) + \sum_{j>0} \frac{j}{r}\operatorname{cof}\Phi_j(r)e_r(j\theta) \otimes e_r(\theta) + \operatorname{cof}\Phi_j'(r)e_\theta(j\theta) \otimes e_\theta(\theta), \tag{2.4}$$

and

$$\int_0^{2\pi} \frac{1}{2}|\nabla\varphi|^2 \, \mathrm{d}\theta = \frac{\pi}{4}\left|\Phi_0'(r)\right|^2 + \frac{\pi}{2}\sum_{j>0}\left|\Phi_j'(r)\right|^2 + \frac{j^2}{r^2}\left|\Phi_j(r)\right|^2, \tag{2.5}$$

$$\int_0^{2\pi} \det\nabla\varphi \, \mathrm{d}\theta = \pi\sum_{j\geqslant 0}\frac{j}{r}(\det\Phi_j(r))'. \tag{2.6}$$

Proof. The formula for $\nabla\varphi$ can be obtained by applying the gradient to each term in (2.1) using

$$\nabla\varphi = \varphi_{,r} \otimes e_r(\theta) + \frac{1}{r}\varphi_{,\theta} \otimes e_\theta(\theta).$$

To get $\operatorname{cof}\nabla\varphi$, we take the cofactor of each term, since it is a linear operator on $\mathbb{R}^{2\times 2}$, and then use

$$\operatorname{cof}(u \otimes v) = (Ju) \otimes (Jv),$$

where $J = R\left(\frac{\pi}{2}\right)$. Then

$$\begin{aligned}
\frac{1}{2}|\nabla\varphi|^2 &= \frac{1}{2}\nabla\varphi\cdot\nabla\varphi \\
&= \frac{1}{2}\left(\sum_{i\geqslant 0} c_i\Phi_i'(r)e_r(i\theta)\otimes e_r(\theta) + \frac{i}{r}\Phi_i(r)e_\theta(i\theta)\otimes e_\theta(\theta)\right) \\
&\cdot\left(\sum_{j\geqslant 0} c_j\Phi_j'(r)e_r(j\theta)\otimes e_r(\theta) + \frac{j}{r}\Phi_j(r)e_\theta(j\theta)\otimes e_\theta(\theta)\right) \\
&= \frac{1}{2}\sum_{i,j\geqslant 0} c_i c_j \left(\Phi_i'(r)e_r(i\theta)\right)\cdot\left(\Phi_j'(r)e_r(j\theta)\right) + \frac{ij}{r^2}\left(\Phi_i(r)e_\theta(i\theta)\right)\cdot\left(\Phi_j(r)e_\theta(j\theta)\right) \\
&= \frac{1}{2}\sum_{i,j\geqslant 0} c_i c_j \left(\Phi_i'(r)e_r(i\theta)\right)\cdot\left(\Phi_j'(r)e_r(j\theta)\right) + \frac{ij}{r^2}\left(\operatorname{cof}\Phi_i(r)e_r(i\theta)\right)\cdot\left(\operatorname{cof}\Phi_j(r)e_r(j\theta)\right),
\end{aligned}$$

and

$$
\begin{aligned}
\det \nabla\varphi &= \frac{1}{2}\nabla\varphi \cdot \operatorname{cof}\nabla\varphi \\
&= \frac{1}{2}\left(\sum_{i\geqslant 0} c_i\Phi_i'(r)e_r(i\theta)\otimes e_r(\theta) + \frac{i}{r}\Phi_i(r)e_\theta(i\theta)\otimes e_\theta(\theta)\right) \\
&\cdot \left(\sum_{j\geqslant 0}\frac{j}{r}\operatorname{cof}\Phi_j(r)e_r(j\theta)\otimes e_r(\theta) + c_j\operatorname{cof}\Phi_j'(r)e_\theta(j\theta)\otimes e_\theta(\theta)\right) \\
&= \frac{1}{2}\sum_{i,j\geqslant 0}\frac{jc_i}{r}\left(\Phi_i'(r)e_r(i\theta)\right)\cdot\left(\operatorname{cof}\phi_j(r)e_r(j\theta)\right) + \frac{ic_j}{r}\left(\Phi_i(r)e_\theta(i\theta)\right)\cdot\left(\operatorname{cof}\Phi_j'(r)e_\theta(j\theta)\right) \\
&= \frac{1}{2}\sum_{i,j\geqslant 0}\frac{jc_i}{r}\left(\Phi_i'(r)e_r(i\theta)\right)\cdot\left(\operatorname{cof}\Phi_j(r)e_r(j\theta)\right) + \frac{ic_j}{r}\left(\operatorname{cof}\Phi_i(r)e_r(i\theta)\right)\cdot\left(\Phi_j'(r)e_r(j\theta)\right),
\end{aligned}
$$

where $c_k := 1 - \frac{\delta_k}{2}$. Here we have used the fact that

$$\operatorname{cof} M = JMJ^{\mathsf{T}} = J^{\mathsf{T}}MJ,$$

which can be checked by direct calculation. We then integrate both over $\theta \in [0, 2\pi)$ using Lemma 2.3 to obtain

$$
\begin{aligned}
\int_0^{2\pi}\frac{1}{2}|\nabla\varphi|^2\,\mathrm{d}\theta &= \frac{\pi}{2}\sum_{i,j\geqslant 0} c_ic_j\left(\delta_{i,j}+\delta_i\delta_j\right)\Phi_i'(r)\cdot\Phi_j'(r) + \frac{ij}{r^2}\left(\delta_{i,j}+\delta_i\delta_j\right)\operatorname{cof}\Phi_i(r)\cdot\operatorname{cof}\Phi_j(r) \\
&= \frac{\pi}{2}\sum_{j\geqslant 0} c_j^2(1+\delta_j)\left|\Phi_j'(r)\right|^2 + \frac{j^2}{r^2}(1+\delta_j)\left|\operatorname{cof}\Phi_j(r)\right|^2 \\
&= \frac{\pi}{2}\sum_{j\geqslant 0} c_j^2(1+\delta_j)\left|\Phi_j'(r)\right|^2 + \frac{j^2}{r^2}(1+\delta_j)\left|\Phi_j(r)\right|^2 \\
&= \frac{\pi}{4}\left|\Phi_0'(r)\right|^2 + \frac{\pi}{2}\sum_{j>0}\left|\Phi_j'(r)\right|^2 + \frac{j^2}{r^2}\left|\Phi_j(r)\right|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{2\pi}\det\nabla\varphi\,\mathrm{d}\theta &= \frac{\pi}{2}\sum_{i,j\geqslant 0}\frac{jc_i}{r}\left(\delta_{i,j}+\delta_i\delta_j\right)\Phi_i'(r)\cdot\operatorname{cof}\Phi_j(r) + \frac{ic_j}{r}\left(\delta_{i,j}+\delta_i\delta_j\right)\operatorname{cof}\Phi_i(r)\cdot\Phi_j'(r) \\
&= \frac{\pi}{2}\sum_{j\geqslant 0}\frac{jc_j}{r}(1+\delta_j)\Phi_j'(r)\cdot\operatorname{cof}\Phi_j(r) + \frac{jc_j}{r}(1+\delta_j)\operatorname{cof}\Phi_j(r)\cdot\Phi_j'(r) \\
&= \frac{\pi}{2}\sum_{j\geqslant 0}\frac{jc_j}{r}(1+\delta_j)(2\det\Phi_j(r))' \\
&= \pi\sum_{j\geqslant 0}\frac{j}{r}(\det\Phi_j(r))'.
\end{aligned}
$$

Note that we can omit the $j = 0$ summand for the determinant as $\det\Phi_0 = 0$ by (2.2). $\square$

By integrating over $\theta \in [0, 2\pi)$, we are essentially taking the leading-order mode, discarding any information that may be contained in higher-order modes. When using Fourier decompositions to obtain sufficient conditions for a non-negative excess, the discarding of higher-order modes will be a significant limitation. However, higher-order modes do not possess the *splitting* behaviour observed in the leading-order mode, and, instead we observe mode *mixing*, making calculations much more difficult. Now that we can Fourier decompose energy terms of maps, we can start to investigate the regularity of the Fourier coefficients of a given map.

Proposition 2.6. *Let $\varphi \in L^2(B;\mathbb{R}^2)$ have Fourier coefficients Φ_j. Then*

$$\int_0^1 r\left|\Phi_j(r)\right|^2 \,\mathrm{d}r < \infty, \qquad \forall j \geqslant 0.$$

Furthermore, if $\varphi \in H^1(B;\mathbb{R}^2)$, then

$$\int_0^1 \frac{1}{r}\left|\Phi_j(r)\right|^2 \,\mathrm{d}r < \infty, \qquad \forall j > 0,$$
$$\int_0^1 r\left|\Phi_j'(r)\right|^2 \,\mathrm{d}r < \infty, \qquad \forall j \geqslant 0.$$

Proof. Similarly to the derivation of (2.5), we start by calculating

$$\begin{aligned}\int_0^{2\pi} |\varphi|^2 \,\mathrm{d}\theta &= \int_0^{2\pi} \sum_{i,j\geqslant 0} c_i c_j \left(\Phi_i(r)e_r(i\theta)\right)\cdot\left(\Phi_j(r)e_r(j\theta)\right)\,\mathrm{d}\theta \\ &= \sum_{i,j\geqslant 0} \pi c_i c_j (\delta_{i,j} + \delta_i\delta_j)\Phi_i(r)\cdot\Phi_j(r) \\ &= \pi\sum_{j\geqslant 0} c_j^2(1+\delta_j)\left|\Phi_j\right|^2 \\ &= \frac{\pi}{2}\left|\Phi_0(r)\right|^2 + \pi\sum_{j>0}\left|\Phi_j(r)\right|^2,\end{aligned}$$

and so

$$\|\varphi\|_2^2 = \int_0^1\int_0^{2\pi} |\varphi|^2\, r\,\mathrm{d}\theta\,\mathrm{d}r = \frac{\pi}{2}\int_0^1 r\left|\Phi_0(r)\right|^2 \,\mathrm{d}r + \pi\sum_{j>0}\int_0^1 r\left|\Phi_j(r)\right|^2 \,\mathrm{d}r.$$

Hence, we require that

$$\int_0^1 r\left|\Phi_j(r)\right|^2 \,\mathrm{d}r < \infty, \qquad \forall j \geqslant 0.$$

Now we further impose $\varphi \in H^1(B;\mathbb{R}^2)$ so $\nabla\varphi \in L^2(B;\mathbb{R}^2)$. Then

$$\|\nabla\varphi\|_2^2 = \int_0^1\int_0^{2\pi} |\nabla\varphi|^2\, r\,\mathrm{d}\theta\,\mathrm{d}r = \frac{\pi}{2}\int_0^1 r\left|\Phi_0'(r)\right|^2 \,\mathrm{d}r + \pi\sum_{j>0}\int_0^1 r\left|\Phi_j'(r)\right|^2 + \frac{j^2}{r}\left|\Phi_j(r)\right|^2 \,\mathrm{d}r.$$

Hence, we require that

$$\int_0^1 \frac{1}{r} \left|\Phi_j(r)\right|^2 \, \mathrm{d}r < \infty, \qquad \forall j > 0,$$
$$\int_0^1 r \left|\Phi_j'(r)\right|^2 \, \mathrm{d}r < \infty, \qquad \forall j \geqslant 0.$$

□

Note that these conditions are necessary but not sufficient. For sufficient conditions, we would need to specify not just that the modes have finite energy but also growth conditions to ensure convergence. One may also wonder if one of the conditions for the case of $\varphi \in H^1$ could be dropped by using an embedding. However, we can show that this is not possible.

Proposition 2.7. *Let $j > 0$. There does not exist a constant $C > 0$ such that*

$$\int_0^1 \frac{1}{r} \left|\Phi_j(r)\right|^2 \, \mathrm{d}r \leqslant C \int_0^1 r \left|\Phi_j'(r)\right|^2 \, \mathrm{d}r, \qquad \forall \varphi \in H^1(B; \mathbb{R}^2). \tag{2.7}$$

Proof. For a counter-example consider the one-parameter family of functions $\psi_\epsilon : [0,1] \to \mathbb{R}$, given by

$$\psi_\epsilon(r) = \begin{cases} \frac{r}{\epsilon}, & r \in [0, \epsilon], \\ 1, & r \in [\epsilon, 1], \end{cases}$$

with $\epsilon \in (0, 1)$.

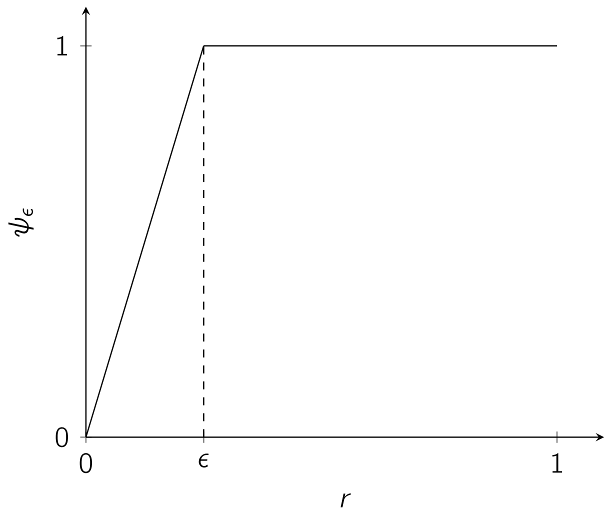

Figure 2.1: Plot of ψ_ϵ for a given $\epsilon > 0$.

We then take $\Phi_j(r) = \psi_\epsilon(r)I$ so we have

$$\int_0^1 \frac{1}{r}\left|\Phi_j(r)\right|^2 \mathrm{d}r = \int_0^\epsilon \frac{2r}{\epsilon^2}\mathrm{d}r + \int_\epsilon^1 \frac{2}{r}\mathrm{d}r = 1 - 2\log(\epsilon),$$
$$\int_0^1 r\left|\Phi_j'(r)\right|^2 \mathrm{d}r = \int_0^\epsilon \frac{2r}{\epsilon^2}\mathrm{d}r + \int_\epsilon^1 0\,\mathrm{d}r = 1.$$

Thus, as $\epsilon \to 0^+$, we have

$$\int_0^1 \frac{1}{r}\left|\Phi_j(r)\right|^2 \mathrm{d}r \to \infty,$$
$$\int_0^1 r\left|\Phi_j'(r)\right|^2 \mathrm{d}r \to 1,$$

which is incompatible with (2.7). □

2.2. Existence of Admissible Maps

Formally, the Euler-Lagrange equation for the excess functional $\mathbb{E}_\mathsf{p}$ is given by

$$\Delta u + \mathrm{cof}(\nabla u)\nabla \mathsf{p} = 0. \tag{2.8}$$

Under the assumption that $\mathsf{p} : B \to \mathbb{R}$ is sufficiently smooth and radially symmetric, we can Fourier decompose the PDE to get an infinite system of ODEs. To do this we must first establish a decomposition for the Laplacian term.

Lemma 2.8. *Let $\varphi : B \to \mathbb{R}^2$ by sufficiently smooth and have Fourier coefficients Φ_j. Then*

$$\Delta\varphi = \frac{1}{2}\left(\Phi_0''(r) + \frac{1}{r}\Phi_0'(r)\right)e_1 + \sum_{j>0}\left(\Phi_j''(r) + \frac{1}{r}\Phi_j'(r) - \frac{j^2}{r^2}\Phi_j(r)\right)e_r(j\theta).$$

Proof. This follows immediately by applying the Laplacian [36] $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$ to each term of the Fourier series of φ. □

We can then derive the following result.

Proposition 2.9. *Let $\mathsf{p} \in C^1(B)$ be radially symmetric. Then (2.8) is equivalent to*

$$r^2A_j''(r) + rA_j'(r) + \left(jr\mathsf{p}'(r) - j^2\right)A_j(r) = 0, \qquad \forall j \geqslant 0, \tag{2.9}$$
$$r^2B_j''(r) + rB_j'(r) - \left(jr\mathsf{p}'(r) + j^2\right)B_j(r) = 0, \qquad \forall j \geqslant 0, \tag{2.10}$$

where $A_j := U_j^{(+)}$ and $B_j := U_j^{(-)}$.

Here we define the conformal and anti-conformal parts of $M \in \mathbb{R}^{2\times 2}$ to be

$$M^{(+)} := \frac{1}{2}\left(M + \operatorname{cof} M\right),$$
$$M^{(-)} := \frac{1}{2}\left(M - \operatorname{cof} M\right),$$

respectively.

Proof. Using (2.8) and (2.4), we already have Fourier decompositions for the Laplacian and cofactor terms. Since p is radially symmetric, we have

$$\nabla \mathsf{p} = \mathsf{p}'(r) e_r(\theta),$$

where the prime denotes a radial derivative. Then, using the fact that

$$(v \otimes a)b = (a \cdot b)v, \qquad \forall a, b, v \in \mathbb{R}^2,$$

we obtain

$$\operatorname{cof}(\nabla u)\nabla \mathsf{p} = \sum_{j>0} \frac{j}{r}\mathsf{p}'(r) \operatorname{cof} U_j(r) e_r(j\theta).$$

Multiplying through by r^2, we find that (2.8) is equivalent to

$$\frac{1}{2}\left(r^2 U_0''(r) + rU_0'(r)\right) e_1 + \sum_{j>0}\left(r^2 U_j''(r) + rU_j'(r) - j^2 U_j(r) + jr\mathsf{p}'(r)\operatorname{cof} U_j(r)\right) e_r(j\theta) = 0,$$

and so

$$r^2 U_j''(r) + rU_j'(r) - j^2 U_j(r) + jr\mathsf{p}'(r)\operatorname{cof} U_j(r) = 0, \qquad \forall j \geqslant 0. \tag{2.11}$$

We now write $U_j = A_j + B_j$ with A_j being the conformal part [22] of U_j and B_j being the anti-conformal [22] part of U_j. Since (2.11) is linear in U_j, taking the conformal and anti-conformal parts yields the same equation but in A_j and B_j, respectively. Then, using the fact that

$$\operatorname{cof} A_j = +A_j,$$
$$\operatorname{cof} B_j = -B_j,$$

the system decouples, giving us

$$r^2 A_j''(r) + rA_j'(r) + \left(jr\mathsf{p}'(r) - j^2\right) A_j(r) = 0, \qquad \forall j \geqslant 0,$$
$$r^2 B_j''(r) + rB_j'(r) - \left(jr\mathsf{p}'(r) + j^2\right) B_j(r) = 0, \qquad \forall j \geqslant 0.$$

□

2.2.1. Monomial Pressure Functions

We observe that (2.9) and (2.10) bear some resemblance to Bessel and modified Bessel equations [1, Chapter 9], respectively, of degree j. In fact, when p takes a certain form, we can show that these ODEs are equivalent to (modified) Bessel equations.

Corollary 2.10. *Suppose, additionally, that $\mathsf{p} : B \to \mathbb{R}$ is smooth away from the origin. The ODEs (2.9) and (2.10) can be transformed into Bessel and modified Bessel equations, respectively, iff p takes the form*

$$\mathsf{p}(r) = \mathsf{p}_0 + \mathsf{p}_1 r^\sigma, \qquad \mathsf{p}_0, \mathsf{p}_1 \in \mathbb{R},$$

for a constant exponent $\sigma > 0$. The corresponding u is given by

$$u = \frac{1}{2}U_0(1)e_1 + \sum_{j>0} \left(\frac{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}}\right)} U_j^{(+)}(1) + \frac{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}}\right)} U_j^{(-)}(1) \right) e_r(j\theta).$$

Proof. The only change of variables that could produce a (modified) Bessel equation takes the form

$$\kappa s^2 = r\mathsf{p}'(r), \qquad \kappa \in \mathbb{R}$$

so

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{r\mathsf{p}''(r) + \mathsf{p}'(r)}{2\kappa s}.$$

The first order differential operator is then given by

$$\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}s}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}s} = \frac{r\mathsf{p}''(r) + \mathsf{p}'(r)}{2\kappa s}\frac{\mathrm{d}}{\mathrm{d}s}.$$

To calculate the second order order differential operator, we use the product rule

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}r^2} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r\mathsf{p}''(r) + \mathsf{p}'(r)}{2\kappa s}\frac{\mathrm{d}}{\mathrm{d}s}\right) \\
&= \left(\frac{r\mathsf{p}''(r) + \mathsf{p}'(r)}{2\kappa s}\right)\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\frac{\mathrm{d}}{\mathrm{d}r}\left(r\mathsf{p}''(r) + \mathsf{p}'(r)\right)(2\kappa s) - \left(r\mathsf{p}''(r) + \mathsf{p}'(r)\right)\frac{\mathrm{d}}{\mathrm{d}r}(2\kappa s)}{(2\kappa s)^2}\frac{\mathrm{d}}{\mathrm{d}s} \\
&= \left(\frac{r\mathsf{p}''(r) + \mathsf{p}'(r)}{2\kappa s}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \left(\frac{r\mathsf{p}'''(r) + 2\mathsf{p}''(r)}{2\kappa s} - \frac{\left(r\mathsf{p}''(r) + \mathsf{p}'(r)\right)^2}{(2\kappa)^2 s^3}\right)\frac{\mathrm{d}}{\mathrm{d}s}.
\end{aligned}$$

Then

$$\begin{aligned} r^2\frac{\mathrm{d}}{\mathrm{d}r} + r\frac{\mathrm{d}}{\mathrm{d}r} &= r^2\left(\frac{r\mathsf{p}''(r)+\mathsf{p}'(r)}{2\kappa s^2}\right)^2 s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} \\ &\quad + \left(r^2\left(\frac{r\mathsf{p}'''(r)+2\mathsf{p}''(r)}{2\kappa s^2} - \frac{(r\mathsf{p}''(r)+\mathsf{p}'(r))^2}{(2\kappa)^2 s^4}\right) + r\left(\frac{r\mathsf{p}''(r)+\mathsf{p}'(r)}{2\kappa s^2}\right)\right) s\frac{\mathrm{d}}{\mathrm{d}s} \\ &= \left(\frac{r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2}\right)^2 s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} \\ &\quad + \left(\frac{r^3\mathsf{p}'''(r)+3r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2} - \left(\frac{r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2}\right)^2\right) s\frac{\mathrm{d}}{\mathrm{d}s}. \end{aligned}$$

If we wish to obtain a (modified) Bessel equation in the variable s, we would require that

$$\left(\frac{r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2}\right)^2 = \frac{r^3\mathsf{p}'''(r)+3r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2} - \left(\frac{r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2}\right)^2 = \frac{\sigma^2}{4} \geqslant 0,$$

for some constant $\sigma > 0$. We start by solving

$$\left(\frac{r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2}\right)^2 = \frac{\sigma^2}{4} \qquad \Longleftrightarrow \qquad r\mathsf{p}''(r) + (1-\sigma)\mathsf{p}'(r) = 0,$$

to obtain

$$\mathsf{p}(r) = \mathsf{p}_0 + \mathsf{p}_1 r^\sigma,$$

for some constants $\mathsf{p}_0, \mathsf{p}_1 \in \mathbb{R}$. It then follows, by direct calculation, that

$$\frac{r^3\mathsf{p}'''(r)+3r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2} - \left(\frac{r^2\mathsf{p}''(r)+r\mathsf{p}'(r)}{2\kappa s^2}\right)^2 = \frac{\sigma^2}{4}.$$

By taking the other sign in the square root, we can also obtain the same solution but with $-\sigma < 0$ instead of $\sigma > 0$. However, this gives a discontinuous pressure function p which we will omit. The ODEs can now be written in the new variable s as

$$\begin{aligned} s^2\tilde{A}_j''(s) + s\tilde{A}_j'(s) + \frac{4}{\sigma^2}\left(j\kappa s^2 - j^2\right)\tilde{A}_j(s) &= 0, \qquad \forall j \geqslant 0, \\ s^2\tilde{B}_j''(s) + s\tilde{B}_j'(s) - \frac{4}{\sigma^2}\left(j\kappa s^2 + j^2\right)B_j(s) &= 0, \qquad \forall j \geqslant 0. \end{aligned}$$

We are free to pick $\kappa \in \mathbb{R}$ for each ODE and so we shall take $\kappa = \frac{\sigma^2}{4j}$, for $j > 0$, to obtain

$$\begin{aligned} s^2\tilde{A}_j''(s) + s\tilde{A}_j'(s) + \left(s^2 - \nu^2\right)\tilde{A}_j(s) &= 0, \qquad \forall j > 0, \\ s^2\tilde{B}_j''(s) + s\tilde{B}_j'(s) - \left(s^2 + \nu^2\right)B_j(s) &= 0, \qquad \forall j > 0, \end{aligned}$$

where $\nu := \frac{2j}{\sigma}$. Thus, when $j > 0$, we have a Bessel equation for the $\tilde{A}_j$ and a modified Bessel equation for the $\tilde{B}_j$, each with degree $\nu = \frac{2\sigma}{j}$. Solving the ODEs in the variable s using the fact that $\tilde{A}_j(s)$ and $\tilde{B}_j(s)$ are bounded near $s = 0$, we find

$$\tilde{A}_j(s) = \frac{J_{\frac{2j}{\sigma}}(s)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}\tilde{A}_j\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right), \qquad j > 0,$$

$$\tilde{B}_j(s) = \frac{I_{\frac{2j}{\sigma}}(s)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}\tilde{B}_j\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right), \qquad j > 0.$$

Now we change back into the variable r to get

$$A_j(r) = \frac{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}A_j(1), \qquad j > 0$$

$$B_j(r) = \frac{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}B_j(1), \qquad j > 0.$$

For $j = 0$, we have

$$r^2 A_0''(r) + rA_0'(r) = 0 \quad \Rightarrow \quad A_0(r) = A_0(1),$$
$$r^2 B_0''(r) + rB_0'(r) = 0 \quad \Rightarrow \quad B_0(r) = B_0(1),$$

again, using the fact that the A_j and B_j are bounded near $r = 0$. Then, since $J_0(0) = I_0(0) = 1$, we can write

$$A_j(r) = \frac{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}A_j(1), \qquad j \geqslant 0$$

$$B_j(r) = \frac{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}B_j(1), \qquad j \geqslant 0.$$

Hence, we can write down a solution for u

$$u = \frac{1}{2}U_0(1)e_1 + \sum_{j>0}\left(\frac{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}U_j^{(+)}(1) + \frac{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)}U_j^{(-)}(1)\right)e_r(j\theta).$$

In the case of $\mathsf{p}_1 < 0$, it may be useful to write

$$\frac{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)} = \frac{I_{\frac{2j}{\sigma}}\left(2\sqrt{-\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{-\frac{j}{\sigma}\mathsf{p}_1}\right)},$$
$$\frac{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{I_{\frac{2j}{\sigma}}\left(2\sqrt{\frac{j}{\sigma}\mathsf{p}_1}\right)} = \frac{J_{\frac{2j}{\sigma}}\left(2\sqrt{-\frac{j}{\sigma}\mathsf{p}_1 r^\sigma}\right)}{J_{\frac{2j}{\sigma}}\left(2\sqrt{-\frac{j}{\sigma}\mathsf{p}_1}\right)},$$

to avoid the introduction of imaginary roots.

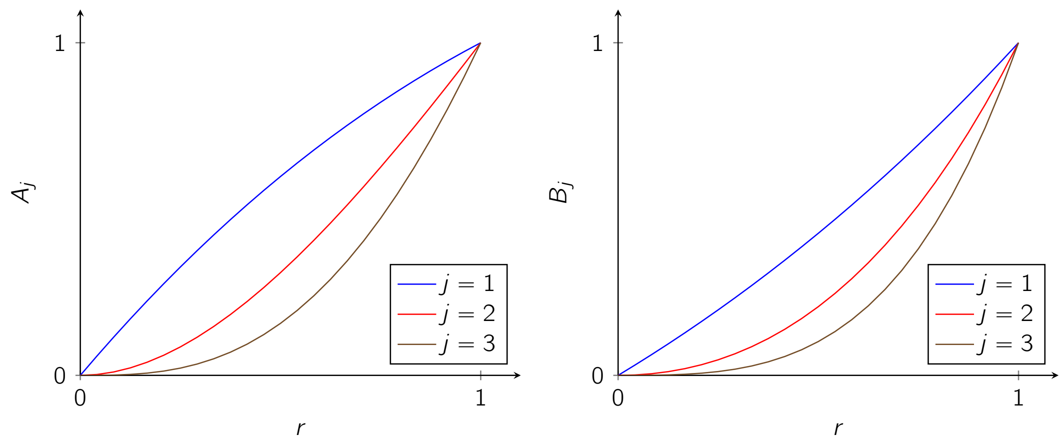

Figure 2.2: Plots of the solutions A_j (left) and B_j (right) with $\sigma = 1$ and $\mathsf{p}_1 = 1$ for $j = 1, 2, 3$. Here we have normalised to $A_j(1) = B_j(1) = 1$.

□

2.2.2. Logarithmic Pressure Functions

We can also observe, in the case of constant (but non-zero) $r\mathsf{p}'(r)$, that (2.9) and (2.10) reduce to Cauchy-Euler equations [11].

Corollary 2.11. *The ODEs (2.9) and (2.10) can be transformed into Cauchy-Euler equations, iff* p *takes the form*

$$\mathsf{p}(r) = \mathsf{p}_0 + \mathsf{p}_* \log(r), \qquad \mathsf{p}_0, \mathsf{p}_* \in \mathbb{R}.$$

A corresponding u exists iff

$$U_j^{(+)}(1) = 0, \qquad \forall j < +\mathsf{p}_*,$$
$$U_j^{(-)}(1) = 0, \qquad \forall j < -\mathsf{p}_*,$$

and is given by

$$u = \frac{1}{2}U_0(1)e_1 + \sum_{j>0}\left(r^{\sqrt{j^2-j\mathsf{p}_*}}U_j^{(+)}(1) + r^{\sqrt{j^2+j\mathsf{p}_*}}U_j^{(-)}(1)\right)e_r(j\theta).$$

Proof. For the ODEs to take the form of a Cauchy-Euler equation, we must have $r\mathsf{p}'(r) = \mathsf{p}_* \in \mathbb{R}$ and changing variables will not make any difference. We then have

$$\mathsf{p}(r) = \mathsf{p}_0 + \mathsf{p}_*\log(r), \qquad \mathsf{p}_0, \mathsf{p}_* \in \mathbb{R}.$$

The ODEs then simplify to

$$r^2 A_j''(r) + rA_j'(r) + \left(j\mathsf{p}_* - j^2\right)A_j(r) = 0, \qquad \forall j \geqslant 0, \tag{2.12}$$

$$r^2 B_j''(r) + rB_j'(r) - \left(j\mathsf{p}_* + j^2\right)B_j(r) = 0, \qquad \forall j \geqslant 0. \tag{2.13}$$

We then break the solutions down into two cases depending on j, assuming $\mathsf{p}_* > 0$.

- If $j \geqslant \mathsf{p}_*$ or $j = 0$, the solutions are given by

  $$A_j(r) = r^{\sqrt{j^2-j\mathsf{p}_*}}A_j(1),$$
  $$B_j(r) = r^{\sqrt{j^2+j\mathsf{p}_*}}B_j(1).$$

  In the case of $j = \mathsf{p}_*$, the general solution for A_j has a logarithmic term that vanishes due to the need for boundedness near $r = 0$.

- If $j < \mathsf{p}_*$ and $j \neq 0$, then there is no real solution for A_j that is bounded near $r = 0$ unless $A_j(1) = 0$. In this case, we have

  $$A_j(r) = 0,$$
  $$B_j(r) = r^{\sqrt{j^2+j\mathsf{p}_*}}B_j(1).$$

If instead $\mathsf{p}_* < 0$, we obtain similar results, but it is B_j that loses real solutions when $j < -\mathsf{p}_*$. The case of $\mathsf{p}_* = 0$ is trivial. Taking into account these existence conditions, the solution for u is given by

$$u = \frac{1}{2}U_0(1)e_1 + \sum_{j>0}\left(r^{\sqrt{j^2-j\mathsf{p}_*}}U_j^{(+)}(1) + r^{\sqrt{j^2+j\mathsf{p}_*}}U_j^{(-)}(1)\right)e_r(j\theta).$$

□

We again recover the constant pressure solution when we set $\mathsf{p}_* = 0$.

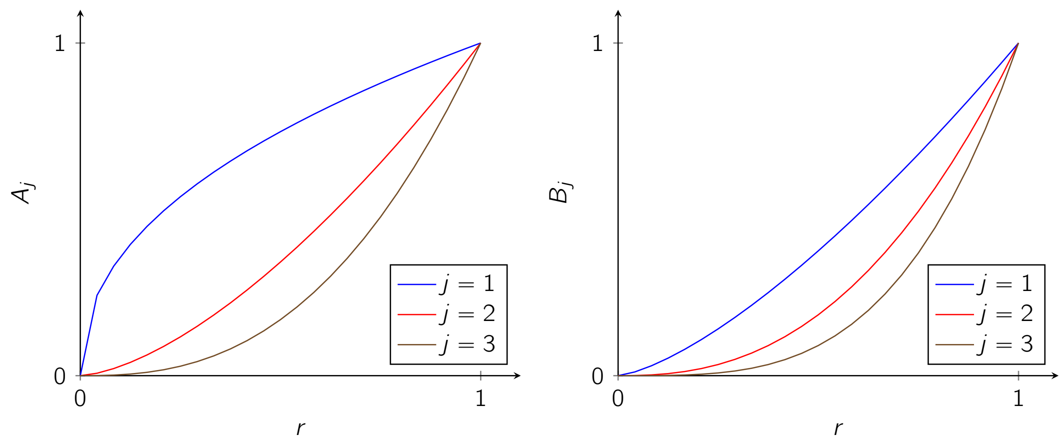

Figure 2.3: Plots of the solutions A_j (left) and B_j (right) with $\mathsf{p}_1 = 0.8$ for $j = 1, 2, 3$. Here we have normalised to $A_j(1) = B_j(1) = 1$.

2.2.3. Frobenius Series Solutions

We now establish a method for constructing admissible maps that correspond to a much wider class of radially symmetric pressure functions, including both the monomial and logarithmic cases previously discussed.

Proposition 2.12. *Let* $\mathsf{p} : B \to \mathbb{R}$ *be radially symmetric and satisfy*

$$\mathsf{p}(r) = \mathsf{p}_* \log(r) + \sum_{k \geqslant 0} \mathsf{p}_k r^k, \qquad \forall r \in (0, 1].$$

Then there exists a solution u *to the Euler-Lagrange equation (2.8) iff*

$$U_j^{(+)}(1) = 0, \qquad \forall j < +\mathsf{p}_*,$$
$$U_j^{(-)}(1) = 0, \qquad \forall j < -\mathsf{p}_*.$$

Proof. Since p has the form

$$\mathsf{p}(r) = \mathsf{p}_* \log(r) + o(\log(r)), \qquad r \to 0^+,$$

we know that $r\mathsf{p}'(r)$ has no pole at $r = 0$. then, by Fuch's theorem (i.e: the Frobenius method) [4, Appendix A], there exist solutions for the A_j and B_j taking the form

$$A_j(r) = \sum_{i \geqslant 0} A_j^i r^{\alpha_j + i}, \qquad A_j^0 \in \{0, 1\}$$
$$B_j(r) = \sum_{i \geqslant 0} B_j^i r^{\beta_j + i}, \qquad B_j^0 \in \{0, 1\}.$$

The condition $A_j^0, B_j^0 \in \{0,1\}$ means that we can have either trivial solutions or non-trivial solutions satisfying a specific boundary condition at $r = 1$. To obtain other non-trivial solutions, we can simply rescale. Substituting these into (2.12) and (2.13)1 gives us

$$\sum_{i \geqslant 0} \left(\left((\alpha_j + i)^2 - j^2 \right) + j(\mathsf{p}_* + o(1)) \right) A_j^i r^{\alpha_j + i} = 0,$$
$$\sum_{i \geqslant 0} \left(\left((\beta + i)^2 - j^2 \right) - j(\mathsf{p}_* + o(1)) \right) B_j^i r^{\beta_j + i} = 0,$$

which have leading order terms

$$\left(\alpha_j^2 - j^2 + j\mathsf{p}_* \right) A_j^i r^{\alpha_j} + o\left(r^{\alpha_j} \right) = 0,$$
$$\left(\beta_j^2 - j^2 - j\mathsf{p}_* \right) B_j^i r^{\beta_j} + o\left(r^{\beta_j} \right) = 0.$$

These leading order terms must vanish for all r and so, depending on the values of j and p_*, we can solve for α_j and β_j.

- Suppose $\mathsf{p}_* \geqslant 0$.
  - If $j \geqslant \mathsf{p}_*$ or $j = 0$, then the indicial equations have roots

    $$\alpha_j = \pm\sqrt{j^2 - j\mathsf{p}_*},$$
    $$\beta_j = \pm\sqrt{j^2 + j\mathsf{p}_*}.$$

    Regardless of whether these roots differ by an integer, we will always discard the solution corresponding to the smaller root, as it will be unbounded near $r = 0$, behaving like either r^γ for some $\gamma < 0$ or $\log(r)\, r^\gamma$ for some $\gamma \leqslant 0$.
  - If $j < \mathsf{p}_*$ and $j \neq 0$, then we still take

    $$\beta_j = \sqrt{j^2 + j\mathsf{p}_*},$$

    but we have no solution for α_j, and so we take $A_j = 0$.
- Now suppose $\mathsf{p}_* \leqslant 0$.
  - If $j \geqslant -\mathsf{p}_*$ or $j = 0$, then the indicial equations have roots

    $$\alpha_j = \pm\sqrt{j^2 - j\mathsf{p}_*},$$
    $$\beta_j = \pm\sqrt{j^2 + j\mathsf{p}_*},$$

    and, again, we discard the solutions corresponding to the smaller roots to ensure boundedness near $r = 0$.

– If $j < -\mathsf{p}_*$ and $j \neq 0$, then we still take

$$\alpha_j = \sqrt{j^2 - j\mathsf{p}_*},$$

but we have no solution for β_j, and so we take $B_j = 0$.

The solutions for the non-trivial A_j and B_j can be obtained using the recurrence relations

$$A_j^i = -\frac{j}{i^2 + 2i\sqrt{j^2 - j\mathsf{p}_*}} \sum_{k+\ell=i} k\mathsf{p}_k A_j^\ell, \qquad A_j^0 = 1, \qquad j \geqslant +\mathsf{p}_* \text{ or } j = 0,$$
$$B_j^i = +\frac{j}{i^2 + 2i\sqrt{j^2 + j\mathsf{p}_*}} \sum_{k+\ell=i} k\mathsf{p}_k B_j^\ell, \qquad B_j^0 = 1, \qquad j \geqslant -\mathsf{p}_* \text{ or } j = 0.$$

□

There are several important remarks that we can make regarding this result.

- The required form for p is dictated entirely by the conditions needed to apply Fuch's theorem. In particular, we require that $\mathsf{p}(r) = \mathsf{p}_* \log(r) + \mu(r)$ where μ is meromorphic [26] and pole-free. However, this immediately tells us that that μ must be expressible as a convergent power series.
- The recurrence relations for the A_j^i and B_j^i are independent of p_0. This is to be expected, as p does not appear explicitly in (2.8), only $\nabla\mathsf{p}$.
- The value of p_* is the main driving factor in the existence of solutions and what boundary conditions we can satisfy. In the absence of a logarithmic term, we can satisfy any boundary condition we like. Furthermore, this also applies when we have a logarithmic term with $|\mathsf{p}_*| \leqslant 1$.

2.2.4. Linear Pressure Functions

The main limitation of the methods discussed so far is the requirement of radial symmetry in the pressure function p. However, (2.8) can also be solved explicitly for some non-radially symmetric pressure functions, one such example being the case of a linear pressure function

$$\mathsf{p}(x) = \lambda \cdot x \qquad \lambda \in \mathbb{R}^2.$$

We can then write the Euler-Lagrange equation as

$$\Delta u + \operatorname{cof}(\nabla u)\lambda = 0. \tag{2.14}$$

This is a coupled linear system with constant coefficients, since $\lambda \in \mathbb{R}^2$ is a constant vector, so we can easily take further derivatives, assuming they exist. We shall take the Laplacian of (2.14) and then substitute for the new Laplacian term, again using (2.14).

$$\begin{aligned}
\Delta^2 u &= -\operatorname{cof}(\nabla \Delta u)\lambda \\
&= -\operatorname{cof}\left(\nabla(-\operatorname{cof}(\nabla u)\lambda)\right)\lambda \\
&= \operatorname{cof}\left(\nabla \begin{pmatrix} -(J\nabla u_2)\cdot\lambda \\ +(J\nabla u_1)\cdot\lambda \end{pmatrix}\right)\lambda \\
&= \operatorname{cof}\left(\nabla \begin{pmatrix} -\nabla u_2 \times \lambda \\ +\nabla u_1 \times \lambda \end{pmatrix}\right)\lambda \\
&= -\begin{pmatrix} \nabla(\nabla u_1 \times \lambda) \times \lambda \\ \nabla(\nabla u_2 \times \lambda) \times \lambda \end{pmatrix} \\
&= -\begin{pmatrix} (\lambda^{\mathsf{T}} J \nabla^2 u_1) \times \lambda \\ (\lambda^{\mathsf{T}} J \nabla^2 u_2) \times \lambda \end{pmatrix} \\
&= -\begin{pmatrix} (J\lambda)\cdot((\nabla^2 u_1)^{\mathsf{T}} J\lambda) \\ (J\lambda)\cdot((\nabla^2 u_2)^{\mathsf{T}} J\lambda) \end{pmatrix} \\
&= -\begin{pmatrix} \operatorname{adj}(\nabla^2 u_1)[\lambda,\lambda] \\ \operatorname{adj}(\nabla^2 u_2)[\lambda,\lambda] \end{pmatrix},
\end{aligned}$$

using the fact that

$$\nabla(v \times \lambda) = -\nabla(v \cdot J\lambda) = -(J\lambda)^{\mathsf{T}}\nabla v = \lambda^{\mathsf{T}} J \nabla v,$$

for constant $\lambda \in \mathbb{R}^2$. We then get the decoupled system

$$\Delta^2 u_j + \operatorname{adj}(\nabla^2 u_j)[\lambda,\lambda] = 0 \qquad j = 1, 2. \tag{2.15}$$

With the assistance of a computer algebra package, we can solve (2.15) and find that solutions u take the form of polynomials with degree 8. In particular, there exists an 18 dimensional solution set to (2.14) consisting of degree 8 polynomials. We will highlight one such example of these polynomial solutions.

Proposition 2.13. *Let $u : B \to \mathbb{R}^2$ be given by*

$$u_j(x) = \left(\frac{\lambda_1}{\lambda_2^2}x_1 + x_2\right)^2 + (-1)^j \frac{2}{\lambda_2}\left(1 + \left(\frac{\lambda_1}{\lambda_2}\right)^2\right)x_1.$$

Then u solves (2.15).

2.3. Sufficient Conditions

Now that we have better established the link between minimising $\mathbb{D}$ amongst the constrained class $\mathcal{A}_u$ and non-negativity of the excess $\mathbb{E}_{\mathsf{p}}$ for a suitable pressure p, we shall proceed to determine which pressure functions make the excess non-negative. Whenever we find a pressure p with $\mathbb{E}_{\mathsf{p}} \geqslant 0$, we can then calculate the corresponding u that solves the Euler-Lagrange equation (2.8) and we know that u will minimise $\mathbb{D}$ in $\mathcal{A}_u$. We shall start by decomposing the excess functional using Fourier modes.

Proposition 2.14. *Let $\mathsf{p} : B \to \mathbb{R}$ be radially symmetric and $\varphi \in H^1(B;\mathbb{R}^2)$. Then*

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \pi \int_0^1 \frac{r}{4}\left|\Phi_0'(r)\right|^2 + \sum_{j>0} \frac{r}{2}\left|\Phi_j'(r)\right|^2 + \frac{j^2}{2r}\left|\Phi_j(r)\right|^2 + j\mathsf{p}(r)\left(\det \Phi_j(r)\right)' \,\mathrm{d}r. \tag{2.16}$$

Furthermore, if p is weakly differentiable, we have

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \pi \int_0^1 \frac{r}{4}\left|\Phi_0'(r)\right|^2 + \sum_{j>0} \frac{r}{2}\left|\Phi_j'(r)\right|^2 + \frac{j^2}{2r}\left|\Phi_j(r)\right|^2 - j\mathsf{p}'(r) \det \Phi_j(r) \,\mathrm{d}r. \tag{2.17}$$

Proof. The proof follows directly from using (2.5) and (2.6), then applying integration by parts. If we estimate the determinant term using Hadamard's inequality and consider Proposition 2.6, we observe that

$$r \mapsto \frac{1}{r} \det \Phi_j(r),$$

is in L^1. Hence, the integral $\int_0^1 \mathsf{p}'(r) \det \Phi_j(r)\,\mathrm{d}r$ is finite if

$$r \mapsto r\mathsf{p}'(r),$$

is in L^∞. □

We observe that the integrand in (2.17) has three terms, namely $\left|\Phi_j'\right|^2$, $\left|\Phi_j\right|^2$ and $\det \Phi_j$. If we can estimate $\left|\Phi_j'\right|^2$ and $\det \Phi_j$ in terms of $\left|\Phi_j\right|^2$, then we can derive a bound of the form

$$\mathbb{E}_{\mathsf{p}}(\varphi) \geqslant \sum_{j\geqslant 0} \int_0^1 \lambda_j(r)\left|\Phi_j(r)\right|^2 \mathrm{d}r, \qquad \forall \varphi \in H_0^1(B;\mathbb{R}^2),$$

where the λ_j depend on p. We can then investigate which choices of p cause the λ_j to be non-negative almost everywhere. To compare $\det \Phi_j$ with $\left|\Phi_j\right|^2$, we will just use Hadamard's inequality [23, 37] pointwise. However, to compare $\left|\Phi_j'\right|^2$ with $\left|\Phi_j\right|^2$, we will need to make use of some form of integral inequality.

2.3.1. Poincaré Inequalities

A natural choice for a mean inequality to compare $\left|\Phi_j'\right|^2$ with $\left|\Phi_j\right|^2$ would be an appropriately weighted Poincaré inequality.

Lemma 2.15. *Let*

$$\mathcal{A} = \left\{ v \in W_0^{1,1}([0,1]) : \int_0^1 r v'(r)^2 \, \mathrm{d}r < \infty \right\}.$$

Then

$$\int_0^1 r v(r)^2 \, \mathrm{d}r \leqslant \frac{1}{j_0^2} \int_0^1 r v'(r)^2 \, \mathrm{d}r, \qquad \forall v \in \mathcal{A},$$

where j_0 is the first positive zero of the Bessel function J_0, and the inequality is sharp.

Proof. We will first minimise the functional

$$F(v) = \int_0^1 r v'(r)^2 \, \mathrm{d}r,$$

in the constrained class

$$\bar{\mathcal{A}} = \left\{ v \in \mathcal{A} : \int_0^1 r v(r)^2 \, \mathrm{d}r = 1 \right\}.$$

We will justify the existence of a minimiser later (see 2.18) and start by calculating a stationary point using the method of Lagrange multipliers [41]. We start by forming the augmented functional

$$\Lambda(v, \lambda) = \int_0^1 r v'(r)^2 \, \mathrm{d}r + \lambda \left(\int_0^1 r v(r)^2 \, \mathrm{d}r - 1 \right),$$

where λ is a Lagrange multiplier. The Euler-Lagrange equation of Λ with respect to λ is just the constraint. However, if we take the Euler-Lagrange equation with respect to v, we have

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(2 r v'(r) \right) = 2 \lambda r v(r),$$

or, equivalently,

$$r^2 v''(r) + r v'(r) - \lambda r^2 v(r) = 0.$$

Making the substitution $s = \sqrt{\lambda} r$ and writing $\tilde{v}(s) = v(r)$, the Euler-Lagrange equation of $\Lambda(\cdot, \lambda)$ can be written as

$$s^2 \tilde{v}''(s) + s \tilde{v}'(s) - s^2 \tilde{v}(s) = 0,$$

which is a Bessel equation of order 0. Thus, the solution is given by

$$\tilde{v}(s) = A J_0(s) + B Y_0(s) \quad \Rightarrow \quad v(r) = A J_0(\sqrt{\lambda} r) + B Y_0(\sqrt{\lambda} r).$$

We then use the boundary conditions to deduce

$$\begin{aligned} v(0) = 0 \quad &\Rightarrow \quad B = 0, \\ v(1) = 0 \quad &\Rightarrow \quad \lambda = j_0^2, \end{aligned}$$

where $j_0 > 0$ is a zero of the Bessel function J_0. Note that we have used that $v \in \bar{\mathcal{A}} \Rightarrow v \not\equiv 0$ in the second line. Furthermore, we have

$$\begin{aligned} 1 = \int_0^1 r v(r)^2 \, \mathrm{d}r = A^2 \int_0^1 r J_0(j_0 r)^2 \, \mathrm{d}r \\ &= \frac{A^2}{j_0^2} \int_0^{j_0} s J_0(s)^2 \, \mathrm{d}s \\ &= \frac{A^2}{j_0^2} \left[\frac{s^2}{2} \left(J_0(s)^2 + J_1(s)^2 \right) \right]_0^{j_0} \\ &= \frac{A^2}{2} J_1(j_0)^2, \end{aligned}$$

and so

$$A = \pm \frac{\sqrt{2}}{J_1(j_0)}.$$

We then substitute this solution back into F to obtain

$$\begin{aligned} F(v) = \int_0^1 r v'(r)^2 \, \mathrm{d}r = \frac{2}{J_1(j_0)^2} \int_0^1 j_0^2 r J_0'(j_0 r)^2 \, \mathrm{d}r \\ &= \frac{2 j_0^2}{J_1(j_0)^2} \int_0^1 r J_1(j_0 r)^2 \, \mathrm{d}r \\ &= \frac{2}{J_1(j_0)^2} \int_0^{j_0} s J_1(s)^2 \, \mathrm{d}s \\ &= \frac{2}{J_1(j_0)^2} \left[\frac{s^2}{2} \left(J_1(s)^2 - J_0(s) J_2(s) \right) \right]_0^{j_0} = j_0^2. \end{aligned}$$

Hence

$$\min_{v \in \bar{\mathcal{A}}} F(v) = j_0^2 \quad \text{at} \quad v^*(r) = \pm \frac{\sqrt{2}}{J_1(j_0)} J_0(j_0 r).$$

Since j_0^2 is the minimum of F in $\bar{\mathcal{A}}$, we will pick j_0 to be the smallest of the positive zeroes of J_0. Now let $v \in \mathcal{A}$ and define

$$\bar{v} := \frac{v}{\sqrt{\int_0^1 r v(r)^2 \, \mathrm{d}r}} \in \bar{\mathcal{A}}.$$

Then

$$F(\bar{v}) = \frac{\int_0^1 r v'(r)^2 \, \mathrm{d}r}{\int_0^1 r v(r)^2 \, \mathrm{d}r} \geqslant j_0^2 \quad \Rightarrow \quad \int_0^1 r v(r)^2 \, \mathrm{d}r \leqslant \frac{1}{j_0^2} \int_0^1 r v'(r)^2 \, \mathrm{d}r \quad \forall v \in \mathcal{A}.$$

The inequality is sharp as equality is attained by taking $v = v^* \in \mathcal{A}$. □

We can now use this weighted Poincaré inequality to estimate the excess functional.

Theorem 2.16. *Let* $\mathsf{p} \in W^{1,1}(B)$ *be radially symmetric and satisfy*

$$\left|r\mathsf{p}'(r)\right| \leqslant \begin{cases} 1 + j_0^2 r^2, & r \in \left[0, \frac{1}{j_0}\right], \\ 2j_0 r & r \in \left[\frac{1}{j_0}, 1\right], \end{cases} \qquad a.e.\ r \in [0,1],$$

Then $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

Proof. We start by recalling the Fourier decomposition of the excess as given in Proposition 2.14.

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \pi \int_0^1 \frac{r}{4}\left|\Phi_0'(r)\right|^2 + \sum_{j>0} \frac{r}{2}\left|\Phi_j'(r)\right|^2 + \frac{j^2}{2r}\left|\Phi_j(r)\right|^2 - j\mathsf{p}'(r)\det\Phi_j(r)\,\mathrm{d}r.$$

By Hadamard's inequality, we have

$$\left|\det\Phi_j(r)\right| \geqslant -\frac{1}{2}\left|\Phi_j(r)\right|^2 \qquad \forall j > 0,$$

pointwise. Then applying Lemma 2.15 to the Φ_j, we estimate

$$\begin{aligned} \mathbb{E}_{\mathsf{p}}(\varphi) &\geqslant \pi \int_0^1 \frac{j_0^2 r}{4}\left|\Phi_0(r)\right|^2 + \sum_{j>0} \frac{j_0^2 r}{2}\left|\Phi_j(r)\right|^2 + \frac{j^2}{2r}\left|\Phi_j(r)\right|^2 - \frac{j}{2}\left|\mathsf{p}'(r)\right|\left|\Phi_j(r)\right|^2\,\mathrm{d}r \\ &= \frac{\pi}{2}\int_0^1 \frac{j_0^2 r}{2}\left|\Phi_0(r)\right|^2 + \sum_{j>0}\left(j_0^2 r + \frac{j^2}{r} - j\left|\mathsf{p}'(r)\right|\right)\left|\Phi_j(r)\right|^2\,\mathrm{d}r. \end{aligned}$$

We can ensure that $\mathbb{E}_{\mathsf{p}} \geqslant 0$ by taking p to satisfy

$$j_0^2 r + \frac{j^2}{r} - j\left|\mathsf{p}'(r)\right| \geqslant 0 \qquad \forall j > 0,$$

pointwise for almost every $r \in [0,1]$. Equivalently, we can write

$$\left|r\mathsf{p}'(r)\right| \leqslant \frac{j_0^2}{j}r^2 + j =: \rho_j(r) \qquad \forall j > 0.$$

To reduce this inequality, we will attempt to minimise the ρ_j with respect to j, observing that the minimum occurs when

$$\frac{\partial\rho_j(r)}{\partial j} = 1 - \frac{j_0^2}{j^2}r^2 = 0 \quad \Rightarrow \quad j = j_0 r.$$

We then split into two possible cases.

- If $r \in \left[\frac{1}{j_0}, 1\right]$, then $j_0 r \geqslant 1$ and so we have

$$\rho_j(r) \geqslant \rho_{j_0 r}(r) = 2j_0 r \qquad \forall j \geqslant 1.$$

- If $r \in \left[0, \frac{1}{j_0}\right]$, then $j_0 r < 1$ and so $\rho_j(r)$ does not attain a local minimum for $j \geqslant 1$. However, we do observe that

$$\frac{\partial\rho_j(r)}{\partial j} = 1 - \frac{j_0^2}{j^2}r^2 \geqslant 1 - \frac{1}{j^2} \geqslant 0,$$

  and so

$$\rho_j(r) \geqslant \rho_1(r) = j_0^2 r^2 + 1 \qquad \forall j \geqslant 1.$$

Overall, we can conclude that

$$\rho_j(r) \geqslant \rho_*(r) := \begin{cases} 1 + j_0^2 r^2, & r \in \left[0, \frac{1}{j_0}\right], \\ 2 j_0 r & r \in \left[\frac{1}{j_0}, 1\right], \end{cases} \qquad \forall j > 0$$

pointwise for $r \in [0, 1]$. Hence, a sufficient condition for $\mathbb{E}_{\mathsf{p}} \geqslant 0$ is

$$\left| r \mathsf{p}'(r) \right| \leqslant \rho_*(r) = \begin{cases} 1 + j_0^2 r^2, & r \in \left[0, \frac{1}{j_0}\right], \\ 2 j_0 r & r \in \left[\frac{1}{j_0}, 1\right], \end{cases} \qquad \text{a.e. } r \in [0, 1]$$

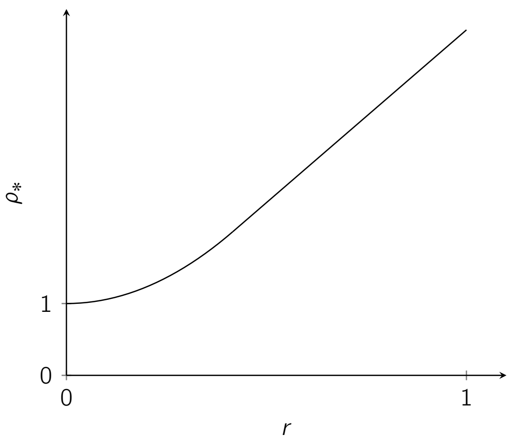

Figure 2.4: Plot of the pointwise upper bound ρ_* derived in Theorem 2.16.

□

As a corollary, we can also deduce a sufficient condition in terms of the Lipschitz constant [8] of the pressure function.

Corollary 2.17. *If* $\mathsf{p} \in W^{1,\infty}(B)$ *is radially symmetric and satisfies*

$$\|\nabla \mathsf{p}\|_\infty \leqslant 2 j_0,$$

then $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

Proof. We first write the sufficient condition as

$$\left| \mathsf{p}'(r) \right| \leqslant \frac{\rho_*(r)}{r},$$

so

$$\left| \mathsf{p}'(r) \right| \leqslant \inf_{s \in [0,1]} \frac{\rho_*(s)}{s} \qquad \text{a.e. } r \in [0, 1],$$

would also be sufficient. Then

$$\frac{\partial}{\partial r}\frac{\rho_*(r)}{r} = \begin{cases} j_0^2 - \frac{1}{r^2}, & r \in \left[0, \frac{1}{j_0}\right], \\ 0 & r \in \left[\frac{1}{j_0}, 1\right], \end{cases}$$

so the minimum is attained at any $r \in \left[\frac{1}{j_0}, 1\right]$ and we have

$$\inf_{r\in[0,1]} \frac{\rho_*(r)}{r} = j_0\rho_*\left(\frac{1}{j_0}\right) = 2j_0.$$

Hence $\|\mathrm{p}'\|_\infty \leqslant 2j_0$ is sufficient for $\mathbb{E}_\mathrm{p} \geqslant 0$. □

It should be noted that the weighted Poincaré inequality in Lemma 2.15 can also be thought of as an unweighted L^2 Poincaré inequality on B, restricted to the class of radially symmetric functions.

$$\int_B |\varphi(x)|^2 \,\mathrm{d}x \leqslant \frac{1}{j_0^2}\int_B |\nabla\varphi(x)|^2 \,\mathrm{d}x \qquad \forall\varphi \in H_0^1(B;\mathbb{R}^2). \tag{2.18}$$

This is why we did not need to show the existence of a minimiser in the proof of Lemma 2.15 as existence of minimisers of the Dirichlet energy, which is the right-hand side of the inequality, has been shown numerous times before [21, Chapter 8.4.3]. We know that solutions to the associated eigenvalue problem for this Poincaré inequality take the form of a Fourier series in polar coordinates and so the principle eigenfunction is radially symmetric. This means that we will see no improvement in the optimal constant when we make the restriction of radial symmetry as the function that makes the inequality sharp is the same in both cases. With this in mind, we can apply the unweighted Poincaré inequality (2.18) directly to $\mathbb{E}_\mathrm{p}$ without any Fourier decomposition.

Proposition 2.18. *Let* $\mathrm{p} \in W^{1,\infty}(B)$ *satisfy*

$$\|\nabla\mathrm{p}\|_\infty \leqslant j_0.$$

Then $\mathbb{E}_\mathrm{p} \geqslant 0$.

Proof. For any $\varphi \in H_0^1(B;\mathbb{R}^2)$, we use Hölder's inequality, Piola's identity and (2.18) to obtain

$$\begin{aligned}
\mathbb{E}_\mathrm{p}(\varphi) &= \int_B \frac{1}{2}|\nabla\varphi|^2 + \mathrm{p}\det\nabla\varphi\,\mathrm{d}x \\
&= \frac{1}{2}\int_B |\nabla\varphi|^2 + \mathrm{p}\,\mathrm{div}\left(\mathrm{cof}(\nabla\varphi)\varphi\right)\,\mathrm{d}x \\
&= \frac{1}{2}\int_B |\nabla\varphi|^2 - \nabla\mathrm{p}\cdot\left(\mathrm{cof}(\nabla\varphi)\varphi\right)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\left(\|\nabla\varphi\|_2^2 - \|\nabla\mathrm{p}\|_\infty\cdot\|\nabla\varphi\|_2\cdot\|\varphi\|_2\right) \\
&\geqslant \frac{1}{2}\left(1 - \frac{\|\nabla\mathrm{p}\|_\infty}{j_0}\right)\|\nabla\varphi\|_2^2.
\end{aligned}$$

Hence,

$$\|\nabla \mathsf{p}\|_\infty \leqslant j_0 \quad \Rightarrow \quad \mathbb{E}_\mathsf{p} \geqslant 0.$$

□

We observe that, when constructing sufficient conditions in terms of Lipschitz constants, making use of Fourier series and assuming radially symmetric pressure give an improvement over direct estimates of the excess functional (since $j_0 < 2j_0$). Furthermore, by using Fourier based techniques, we are able to derive sufficient conditions given in terms of pointwise bounds when the pressure is radially symmetric. This allows us to find non-negative excess functionals for a much wider range of pressure functions, including non-Lipschitz examples.

2.3.2. Weighted Sobolev Inequalities

The main limitation of using Poincaré inequalities in this context is due to the weight being the same on both sides of the inequality. If we look closer at the integrand in (2.17), we don't just want to compare $\left|\Phi_j'\right|^2$ with $\left|\Phi_j\right|^2$, but rather we wish to compare $r\left|\Phi_j'\right|^2$ with $\frac{1}{r}\left|\Phi_j\right|^2$. This motivates the usage of weighted Sobolev inequalities [3, Chapter 1], taking the form

$$\int_B |\varphi(x)|^2 \frac{\mathrm{d}x}{w(x)^2} \leqslant \frac{1}{C_w^2}\int_B |\nabla\varphi(x)|^2 \,\mathrm{d}x, \qquad \forall \varphi \in H_0^1(B;\mathbb{R}^2) \tag{2.19}$$

for a non-negative weight function $w : B \to \mathbb{R}$ that we will assume is radially symmetric. Note that the constant $C_w > 0$ depends on the choice of w, if it exists at all. As we have seen in the case of the Poincaré inequality, it will also be useful to consider a radially symmetric version of this inequality, given by

$$\int_0^1 v(r)^2 \frac{r\,\mathrm{d}r}{w(r)^2} \leqslant \frac{1}{C_w^2}\int_0^1 v'(r)^2 r\,\mathrm{d}r. \tag{2.20}$$

If we can find a weight w and a constant C_w such that we have a weighted Sobolev inequality, we can then use it to derive sufficient conditions for non-negativity of $\mathbb{E}_\mathsf{p}$.

Proposition 2.19. *Let $w : B \to \mathbb{R}$ and $C_w \in \mathbb{R}$ be non-negative and picked such that (2.19) holds for all admissible φ. Then*

$$\|w\nabla \mathsf{p}\|_\infty \leqslant C_w \quad \Rightarrow \quad \mathbb{E}_\mathsf{p} \geqslant 0.$$

Proof. We shall apply the same technique as in the proof of Proposition 2.18, writing

$$\begin{aligned}
\mathbb{E}_\mathsf{p}(\varphi) &= \frac{1}{2}\int_B |\nabla\varphi|^2 - w\nabla\mathsf{p}\cdot\left(\mathrm{cof}(\nabla\varphi)\frac{\varphi}{w}\right)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\left(\|\nabla\varphi\|_2^2 - \|w\nabla\mathsf{p}\|_\infty\cdot\|\nabla\varphi\|_2\cdot\left\|\frac{\varphi}{w}\right\|_2\right) \\
&\geqslant \frac{1}{2}\left(1-\frac{1}{C_w}\|w\nabla\mathsf{p}\|_\infty\right)\|\nabla\varphi\|_2^2 .
\end{aligned}$$

□

We again observe that, when we do not apply a Fourier decomposition to the excess functional, we obtain only a sufficient condition on p in the mean. We can improve this result by using the decomposition technique as demonstrated in the proof of Theorem (2.16).

Theorem 2.20. *Let $w : [0,1] \to \mathbb{R}$ and $C_w \in \mathbb{R}$ be non-negative and picked such that (2.19) holds for all admissible v. If $\mathsf{p} \in W^{1,1}(B)$ is radially symmetric and satisfies*

$$\left|r\mathsf{p}'(r)\right| \leqslant \begin{cases} 1 + \frac{C_w^2}{w(r)^2} r^2, & r \leqslant \frac{w(r)}{C_w}, \\ \frac{2C_w}{w(r)} r, & r \geqslant \frac{w(r)}{C_w}, \end{cases} \qquad a.e.\ r \in [0,1],$$

then $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

Proof. We start as we did in the proof of Theorem (2.16), writing

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \pi \int_0^1 \frac{r}{4} \left|\Phi_0'(r)\right|^2 + \sum_{j>0} \frac{r}{2} \left|\Phi_j'(r)\right|^2 + \frac{j^2}{2r} \left|\Phi_j(r)\right|^2 - j\mathsf{p}'(r) \det \Phi_j(r) \, \mathrm{d}r.$$

We again apply Hadamard's inequality but this time, instead of using Lemma (2.15), we apply (2.20) and obtain

$$\begin{aligned} \mathbb{E}_{\mathsf{p}}(\varphi) &\geqslant \pi \int_0^1 \frac{C_w^2 r}{4w(r)^2} \left|\Phi_0(r)\right|^2 + \sum_{j>0} \frac{C_w^2 r}{2w(r)^2} \left|\Phi_j(r)\right|^2 + \frac{j^2}{2r} \left|\Phi_j(r)\right|^2 - \frac{j}{2} \left|\mathsf{p}'(r)\right| \left|\Phi_j(r)\right|^2 \mathrm{d}r \\ &= \frac{\pi}{2} \int_0^1 \frac{C_w^2 r}{2w(r)^2} + \sum_{j>0} \left(\frac{C_w^2}{w(r)^2} r^2 + j^2 - j\left|r\mathsf{p}'(r)\right| \right) \left|\Phi_j(r)\right|^2 \frac{\mathrm{d}r}{r}. \end{aligned}$$

To ensure $\mathbb{E}_{\mathsf{p}} \geqslant 0$, we take p to satisfy

$$\left|r\mathsf{p}'(r)\right| \leqslant \frac{C_w^2}{jw(r)^2} r^2 + j =: \rho_j(r),$$

pointwise for almost every $r \in [0,1]$ and for each $j > 0$. As we did in the proof of Theorem 2.16, we minimise with respect to $j \geqslant 1$, observing that

$$\frac{\partial \rho_j(r)}{\partial j} = 1 - \frac{C_w^2}{j^2 w(r)^2} r^2 = 0 \quad \Rightarrow \quad j = \frac{C_w r}{w(r)}.$$

As before, we have two possible cases to consider.

- If $r \geqslant \frac{1}{C_w} w(r)$, then $\frac{C_w r}{w(r)} \geqslant 1$ and so

$$\rho_j(r) \geqslant \rho_{\frac{C_w r}{w(r)}}(r) = \frac{2C_w}{w(r)} r \qquad \forall j \geqslant 1.$$

- On the other hand, if $r < \frac{1}{C_w} w(r)$, then $\frac{C_w r}{w(r)} < 1$ and so ρ_j does not attain a local minimiser for $j \geqslant 1$ as it is increasing in j. Hence,

$$\rho_j(r) \geqslant \rho_1(r) = 1 + \frac{C_w^2}{w(r)^2} r^2 \qquad \forall j \geqslant 1.$$

We can write all of this succinctly as

$$\rho_j(r) \geqslant \rho_*(r) := \begin{cases} 1 + \frac{C_w^2}{w(r)^2} r^2, & r \leqslant \frac{w(r)}{C_w}, \\ \frac{2C_w}{w(r)} r, & r \geqslant \frac{w(r)}{C_w}, \end{cases} \qquad \forall j \geqslant 1.$$

We conclude by observing that $|r\mathsf{p}'(r)| \leqslant \rho_*(r)$ pointwise for almost every $r \in [0,1]$ is sufficient for $\mathbb{E}_{\mathsf{p}} \geqslant 0$. □

Note that both Proposition 2.18 and Theorem 2.16 follow as direct corollaries from Proposition 2.19 and Theorem 2.20, respectively, with $w(x) = 1$ and $C_w = j_0$. We also observe that we can normalise the threshold function ρ_* relative to the weight by writing $\rho_*(r) = \mu\left(\frac{C_w r}{w(r)}\right)$ where $\mu : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is independent of the weight function and is given by

$$\mu(t) = \begin{cases} 1 + t^2, & t \leqslant 1, \\ 2t, & t \geqslant 1. \end{cases}$$

As we did with the Poincaré inequality, we can use the pointwise sufficient condition in Theorem 2.20 to derive a sufficient condition in the mean.

Corollary 2.21. *Let $w : [0,1] \to \mathbb{R}$ and $C_w \in \mathbb{R}$ be non-negative and picked such that (2.19) holds for all admissible v. If $\mathsf{p} : B \to \mathbb{R}$ is radially symmetric, then*

$$\|w\nabla \mathsf{p}\|_\infty \leqslant 2C_w \quad \Rightarrow \quad \mathbb{E}_{\mathsf{p}} \geqslant 0.$$

Proof. We first observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu(t)}{t} = \begin{cases} 1 - \frac{1}{t^2}, & t \leqslant 1, \\ 0, & t \geqslant 1, \end{cases}$$

so $t \mapsto \frac{\mu(t)}{t}$ is a continuous function that is strictly decreasing for $t \leqslant 1$ and constant for $t \geqslant 1$, with a minimum of 2. Setting $t = \frac{C_w r}{w(r)}$, we find that

$$\frac{w(r)}{C_w r}\mu\left(\frac{C_w r}{w(r)}\right) \geqslant 2 \qquad \forall r \in [0,1].$$

Using Theorem 2.20, we then have that a sufficient condition for $\mathbb{E}_{\mathsf{p}} \geqslant 0$ is

$$\frac{w(r)}{C_w r}\left|r\mathsf{p}'(r)\right| \leqslant 2,$$

or, equivalently,

$$\|w\nabla \mathsf{p}\|_\infty \leqslant 2C_w.$$

□

As we did in the case of the Poincaré inequality, we observe a doubling of the constant on the right-hand side when we restrict to radially symmetric pressure. Furthermore, we can rescale the constant C_w so that $\|w\|_\infty = 1$ and then we get another sufficient condition for $\mathbb{E}_{\mathbf{p}} \geqslant 0$, given by

$$\|\nabla \mathbf{p}\|_\infty \leqslant (1+\delta) C_w,$$

where we take $\delta = 0$ for the case of a general pressure function and $\delta = 1$ for the radially symmetric case. Thus the mean condition is improved purely by picking w such that the constant C_w is larger. If the pressure is not Lipschitz but is radially symmetric, the pointwise condition given in Theorem 2.20 may still apply.

Not every weight function will give rise to a valid weighted Sobolev inequality. Furthermore, we can derive necessary conditions on the weight w for such an inequality to exist.

Lemma 2.22. *For (2.19) and (2.20) to hold, we require that* $\frac{1}{w} \in L^2(B)$.

Proof. Consider the function $\psi_\epsilon : [0,1] \to \mathbb{R}$ as given in the proof of Proposition 2.7 and again take $\Phi_j(r) = \psi_\epsilon(r) I$. We then have

$$\int_0^1 |\Phi_j(r)|^2 \frac{r\,\mathrm{d}r}{w(r)^2} = \int_0^\epsilon \frac{r^3}{\epsilon^2 w(r)^2}\,\mathrm{d}r + \int_\epsilon^1 \frac{r}{w(r)^2}\,\mathrm{d}r,$$
$$\int_0^1 |\Phi_j'(r)^2|\, r\,\mathrm{d}r = \int_0^\epsilon \frac{2r}{\epsilon^2}\,\mathrm{d}r + \int_\epsilon^1 0\,\mathrm{d}r = 1.$$

Using (2.20), we have

$$\int_0^1 |\Phi_j(r)|^2 \frac{r\,\mathrm{d}r}{w(r)^2} \leqslant \frac{1}{C_w^2} \qquad \forall \epsilon > 0.$$

In particular,

$$\int_\epsilon^1 \frac{r\,\mathrm{d}r}{w(r)^2} \leqslant \frac{1}{C_w^2} \qquad \forall \epsilon > 0.$$

We then apply the monotone convergence theorem [27, 3.3.3] to

$$f_n(r) := \frac{r}{w(r)^2} \chi_{[\frac{1}{n},1]}(r),$$

to find $\frac{r}{w(r)^2} \in L^1(0,1)$, so $\frac{1}{w} \in L^2(B)$.

□

This rules out using $w(r) = r$ as a weight. This is quite unfortunate as this would be the ideal choice of weight when working with the Fourier decomposed excess functional due to the appearance of the $\frac{1}{r}|\Phi_j(r)|^2$ and $r\left|\Phi_j'(r)\right|^2$ terms.

2.3.3. Hardy Inequalities

We now consider one such example of a weighted Sobolev inequality, sometimes referred to as a Hardy inequality [33], where the weight is given by $w(r) = r^{1-\alpha}$ for some $\alpha > 0$.

Lemma 2.23. *Let $\alpha > 0$ and $\mathcal{A}$ be given as in Lemma 2.15. Then*

$$\int_0^1 v(r)^2 r^{2\alpha-1}\,\mathrm{d}r \leqslant \frac{{}_2F_3\left(\frac{1}{2},\alpha;1,1,1+\alpha;-j_0^2\right)}{\alpha j_0^2 J_1(j_0)^2}\int_0^1 v'(r)^2\,\mathrm{d}r \qquad \forall v \in \mathcal{A},$$

and the inequality is sharp.

Proof. We apply the same method as used in the proof of Lemma 2.15, constructing a constrained class

$$\bar{\mathcal{A}} = \left\{ v \in \mathcal{A} : \int_0^1 v(r)^2 r^{2\alpha-1}\,\mathrm{d}r = 1 \right\},$$

and finding stationary functions v of the augmented functional (with Lagrange multiplier λ) that solve

$$r^2 v''(r) + r v'(r) - \lambda r^{2\alpha} v(r) = 0, \qquad v(0) = v(1) = 0. \tag{2.21}$$

We apply a change of variables given by $s = \frac{\sqrt{\lambda}}{\alpha} r^\alpha$ and write $\tilde{v}(s) = v(r)$, to obtain

$$s^2 \tilde{v}''(s) + s\tilde{v}'(s) - s^2 \tilde{v}(s) = 0, \qquad \tilde{v}\,(0) = \tilde{v}\left(\frac{\sqrt{\lambda}}{\alpha}\right) = 0.$$

This is a Bessel equation of degree 0 so we get non-trivial solutions when $\lambda = j_0^2\alpha^2$ and they are given by

$$\tilde{v}(s) = AJ_0(s) \quad \Rightarrow \quad v(r) = AJ_0(j_0 r),$$

for a constant $A \in \mathbb{R}$ to be determined. We then calculate

$$1 = \int_0^1 v(r)^2 r^{2\alpha-1}\,\mathrm{d}r = \frac{A^2}{2\alpha}{}_2F_3\left(\frac{1}{2},\alpha;1,1,1+\alpha;-j_0^2\right),$$

and

$$C_\alpha^2 = \int_0^1 v'(r)^2 r\,\mathrm{d}r = \frac{A^2}{2} j_0^2 J_1(j_0)^2 = \frac{\alpha j_0^2 J_1(j_0)^2}{{}_2F_3\left(\frac{1}{2},\alpha;1,1,1+\alpha;-j_0^2\right)} \sim 2 j_0 J_1(j_0)^2 \alpha \qquad (\alpha \to 0^+),$$

where $\frac{1}{C_\alpha^2}$ is the optimal constant for the inequality on $\bar{\mathcal{A}}$ and ${}_2F_3\,(\cdot)$ is a hypergeometric function [2, Chapter 2]. By the reasoning given in the proof of Lemma 2.15, the inequality also holds on $\mathcal{A}$ and is sharp. □

Note that we recover the weighted Poincaré inequality as seen in Lemma 2.15 by taking $\alpha = 1$. We also observe that the optimal constant $\frac{1}{C_\alpha}$ diverges as $\alpha \to 0^+$, which agrees with the observations of Proposition 2.7 and Lemma 2.22. Furthermore, in the case of $\alpha = 0$, there would be no non-trivial solutions to (2.21). We can use these inequalities to construct new sufficient conditions for non-negativity of the excess functional.

Theorem 2.24. *Let* $\mathrm{p} \in W^{1,1}(B; \mathbb{R}^2)$ *be radially symmetric and satisfy*

$$\left|r\mathrm{p}'(r)\right| \leqslant \begin{cases} 1 + C_\alpha^2 r^{2\alpha}, & r \in \left[0, C_\alpha^{-\frac{1}{\alpha}}\right], \\ 2C_\alpha r^\alpha & r \in \left[C_\alpha^{-\frac{1}{\alpha}}, 1\right], \end{cases} \qquad a.e.\ r \in [0,1]$$

Then $\mathbb{E}_\mathrm{p} \geqslant 0$.

Proof. This result follows as a direct corollary to Theorem 2.20 with $w(r) = r^{1-\alpha}$ and $C_w = C_\alpha$, noting that

$$r \leqslant \frac{C_w}{w(r)} \quad \Longleftrightarrow \quad r \leqslant \frac{1}{C_\alpha^\alpha},$$
$$r \geqslant \frac{C_w}{w(r)} \quad \Longleftrightarrow \quad r \geqslant \frac{1}{C_\alpha^\alpha}.$$

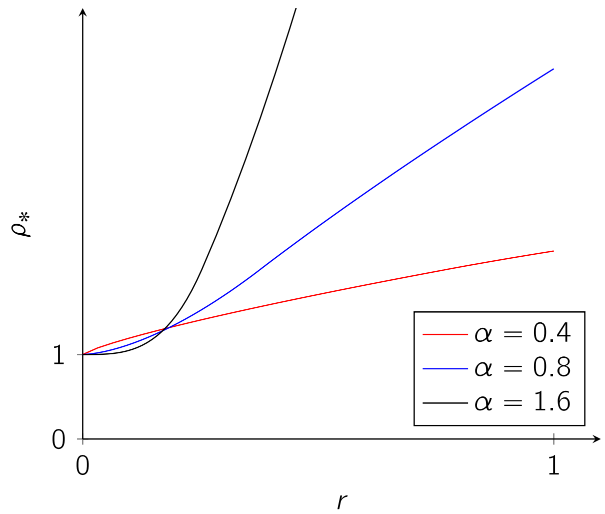

Figure 2.5: Plot of the pointwise upper bound ρ_* derived in Theorem 2.24 for various values of α.

□

As we did in the case of the weighted Poincaré inequality, we can also derive a sufficient condition in terms of the Lipschitz constant of the pressure function.

Corollary 2.25. *If* $\mathrm{p} \in W^{1,\infty}(B)$ *is radially symmetric and satisfies*

$$\|\nabla \mathrm{p}\|_\infty \leqslant \begin{cases} 1 + C_\alpha^2, & \alpha \in (0, \alpha^*], \\ 2C_\alpha, & \alpha \in [\alpha^*, 1], \\ \frac{2\alpha}{(2\alpha-1)^{2\alpha-1}} C_\alpha^{\frac{1}{\alpha}}, & \alpha \in [1, \infty). \end{cases}$$

then $\mathbb{E}_\mathrm{p} \geqslant 0$*, where* $\alpha^* \approx 0.364$ *is the unique solution to* $C_\alpha = 1$.

Proof. We first write the sufficient condition as

$$\left|\mathsf{p}'(r)\right| \leqslant \frac{\rho_*(r)}{r},$$

so

$$\left|\mathsf{p}'(r)\right| \leqslant \inf_{s\in[0,1]} \frac{\rho_*(s)}{s} \qquad \text{a.e. } r \in [0,1],$$

would also be sufficient. Then

$$\frac{\partial}{\partial r}\frac{\rho_*(r)}{r} = \begin{cases} (2\alpha-1)C_\alpha^2 r^{2\alpha-2} - r^{-2}, & r \in \left[0, C_\alpha^{-\frac{1}{\alpha}}\right], \\ (2\alpha-2)C_\alpha r^{\alpha-2} & r \in \left[C_\alpha^{-\frac{1}{\alpha}}, 1\right]. \end{cases}$$

We now split into cases based on whether α is smaller or larger than $\alpha^* < 1$.

- If $\alpha \geqslant 1$ then a local minimum is attained at

$$r = \left(\frac{1}{2\alpha-1}\right)^{\frac{1}{2\alpha}} C_\alpha^{-\frac{1}{\alpha}} \in \left(0, C_\alpha^{-\frac{1}{\alpha}}\right].$$

  Thus,

$$\frac{\rho_*(r)}{r} \geqslant (2\alpha-1)^{\frac{1}{2\alpha}} C_\alpha^{-\frac{1}{\alpha}} \rho_*\left(\left(\frac{1}{2\alpha-1}\right)^{\frac{1}{2\alpha}} C_\alpha^{-\frac{1}{\alpha}}\right) = \frac{2\alpha}{(2\alpha-1)^{2\alpha-1}} C_\alpha^{\frac{1}{\alpha}}.$$

- If $\alpha < 1$, then $\frac{\rho_*}{r}$ is decreasing on $[0,1]$ and

$$\frac{\rho_*(r)}{r} \geqslant \frac{\rho_*(1)}{1} = \begin{cases} 1 + C_\alpha^2, & \alpha \leqslant \alpha^*, \\ 2C_\alpha, & \alpha \geqslant \alpha^*. \end{cases}$$

Overall, we have that

$$\inf_{r\in[0,1]} \frac{\rho_*(r)}{r} = \begin{cases} 1 + C_\alpha^2, & \alpha \in (0, \alpha^*], \\ 2C_\alpha, & \alpha \in [\alpha^*, 1], \\ \frac{2\alpha}{(2\alpha-1)^{2\alpha-1}} C_\alpha^{\frac{1}{\alpha}}, & \alpha \in [1, \infty). \end{cases}$$

□

We can then compare with the sufficient condition we obtain for non-radially symmetric pressure functions.

Proposition 2.26. *Let* $\mathsf{p} \in W^{1,1}(B)$ *satisfy*

$$\left\| |x|^{1-\alpha} \nabla \mathsf{p} \right\|_\infty \leqslant \frac{j_0 J_1(j_0)\sqrt{\alpha}}{\sqrt{{}_2F_3\left(\frac{1}{2}, \alpha; 1, 1, 1+\alpha; -j_0^2\right)}},$$

for some $\alpha > 0$. *Then* $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

Proof. This result also follows as a direct corollary to Theorem 2.20 with $w(x) = |x|^{1-\alpha}$ and $C_w = C_\alpha$. □

Whereas Proposition 2.18 applied only to Lipschitz pressure functions, Proposition 2.26 can be applied to non-Lipschitz pressure functions (when $0 < \alpha < 1$) but fails to be applicable to logarithmic pressure functions, such as $\mathrm{p}(x) = \log(|x|)$. By contrast, Theorem 2.24 can still be applied to logarithmic pressure functions, as we will see later.

2.3.4. The Critical Case

In the limit $\alpha \to 0^+$ the Hardy inequality fails and so we do not have an inequality that can compare $r\left|\Phi_j'\right|^2$ with $\frac{1}{r}\left|\Phi_j\right|^2$ in the Fourier decomposed excess functional. However, there does exist a critical version of the Hardy inequality [35] that replaces the case of $\alpha = 0$.

Lemma 2.27. *Let $\alpha > 0$ and $\mathcal{A}$ be given as in Lemma 2.15. Then*

$$\int_0^1 v(r)^2 \frac{\mathrm{d}r}{r(1-\log(r))^2} \leqslant 4\int_0^1 v'(r)^2\,\mathrm{d}r \qquad \forall v \in \mathcal{A}.$$

This inequality is, in some sense, the best inequality to bound the integrand in (2.17) below purely in terms of the $\left|\Phi_j\right|^2$.

Theorem 2.28. *Let $\mathrm{p} \in W^{1,1}(B;\mathbb{R}^2)$ be radially symmetric and satisfy*

$$\left|r\mathrm{p}'(r)\right| \leqslant 1 + \frac{1}{4(1-\log(r))^2} \qquad a.e.\ r \in [0,1],$$

Then $\mathbb{E}_\mathrm{p} \geqslant 0$.

Proof. We apply Theorem 2.20 with $w(r) = r(1-\log(r))$ and $C_w = \frac{1}{2}$. Here we note that, for $r \in [0,1]$, we have

$$\frac{w(r)}{C_w} = 2r(1-\log(r)) \geqslant 2r \geqslant r,$$

so we need only consider the first case for ρ_*.

□

Once again, we can derive a sufficient condition in terms of the Lipschitz constant of the pressure function.

Corollary 2.29. *If $\mathrm{p} \in W^{1,\infty}(B)$ is radially symmetric and satisfies*

$$\|\nabla \mathrm{p}\|_\infty \leqslant \frac{5}{4}$$

then $\mathbb{E}_\mathrm{p} \geqslant 0$.

Proof. We rewrite the sufficient condition as

$$\left|\mathsf{p}'(r)\right| \leqslant \inf_{s\in[0,1]} \frac{\rho_*(s)}{s} \qquad \text{a.e. } r \in [0,1].$$

Since $\frac{\rho_*(r)}{r}$ is decreasing in r, we have

$$\inf_{r\in[0,1]} \frac{\rho_*(r)}{r} = \frac{\rho_*(1)}{1} = 1 + \frac{1}{4} = \frac{5}{4}.$$

□

As we have done for the Poincaré and Hardy inequalities, we can also derive a weaker result for non-radially symmetric pressure functions.

Proposition 2.30. *Let* $\mathsf{p} \in W^{1,1}(B)$ *satisfy*

$$\left\| |x| \left(1 - \log\left(|x|\right)\right) \nabla \mathsf{p} \right\|_\infty \leqslant \frac{1}{2}.$$

Then $\mathbb{E}_\mathsf{p} \geqslant 0$.

Proof. This follows as a direct application of Proposition 2.19 with $w(x) = |x| \left(1 - \log\left(|x|\right)\right)$ and $C_w = \frac{1}{2}$. □

Again, we see that the radial version gives improved sufficient conditions over the non-radial version. We also note that there exists another form of critical Hardy inequality [41], given by

$$\int_B v(x)^2 \frac{\mathrm{d}x}{|x|^2} \leqslant 4 \int_B |\nabla v(x)|^2 \left(\log\left(|x|\right)\right)^2 \mathrm{d}x,$$

along with a corresponding radial version. However, this form of inequality is of no use in bounding the excess using the techniques provided here due to the incorrect weight on the right-hand side.

2.4. Examples

We shall now use the results of this section to derive some concrete sufficient conditions for non-negativity of the excess functional when the pressure function takes certain forms.

2.4.1. Monomial Pressure Functions

Consider a radially symmetric pressure function $\mathsf{p} : B \to \mathbb{R}^2$ of the form

$$\mathsf{p}(r) = \mathsf{p}_0 + \mathsf{p}_1 r^\sigma \qquad \sigma > 0.$$

We will start by using Theorem 2.16 to derive a sufficient condition for $\mathbb{E}_\mathsf{p} \geqslant 0$ in terms of the value of p_1.

Proposition 2.31. *Let* p_1 *satisfy*

$$|\mathsf{p}_1| \leqslant \begin{cases} 2j_0^\sigma \sigma^{-1-\frac{\sigma}{2}}(2-\sigma)^{-1+\frac{\sigma}{2}}, & \sigma \in (0,1], \\ 2j_0 & \sigma \in [1,\infty). \end{cases}$$

Then $\mathbb{E}_{\mathsf{p}} \geqslant 0$.

Proof. We first calculate

$$\left|r\mathsf{p}'(r)\right| = \sigma\,|\mathsf{p}_1|\,r^\sigma,$$

so, by Theorem 2.16, the excess is non-negative if

$$\sigma\,|\mathsf{p}_1| \leqslant \begin{cases} r^{-\sigma} + j_0^2 r^{2-\sigma}, & r \in \left[0, \frac{1}{j_0}\right], \\ 2j_0 r^{1-\sigma} & r \in \left[\frac{1}{j_0}, 1\right], \end{cases} \qquad \text{a.e. } r \in [0,1].$$

We then seek to minimise the right-hand side with respect to r.

- First consider $r \in \left[0, \frac{1}{j_0}\right]$. Then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-\sigma} + j_0^2 r^{2-\sigma}\right) = -\sigma r^{-\sigma-1} + j_0^2(2-\sigma)r^{1-\sigma} = r^{-\sigma-1}\left(j_0^2(2-\sigma)r^2 - \sigma\right).$$

  Thus, if $\sigma \leqslant 1$, we have a local minima at

$$r = \sqrt{\frac{\sigma}{j_0^2(2-\sigma)}} \quad \Rightarrow \quad r^{-\sigma} + j_0^2 r^{2-\sigma} = \frac{2j_0^\sigma}{2-\sigma}\left(\frac{2-\sigma}{\sigma}\right)^{\frac{\sigma}{2}}.$$

  However, if $\sigma \geqslant 1$, $r \mapsto r^{-\sigma} + j_0^2 r^{2-\sigma}$ is decreasing on $\left[0, \frac{1}{j_0}\right]$ and so the minimum occurs at

$$r = \frac{1}{j_0} \quad \Rightarrow \quad r^{-\sigma} + j_0^2 r^{2-\sigma} = 2j_0^\sigma.$$

- Now we consider $r \in \left[\frac{1}{j_0}, 1\right]$. Here we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(2j_0 r^{1-\sigma}\right) = 2j_0(1-\sigma)r^{-\sigma}.$$

  Hence, $r \mapsto 2j_0 r^{1-\sigma}$ is increasing for $\sigma \leqslant 1$ and so we have a minimum at

$$r = \frac{1}{j_0} \quad \Rightarrow \quad 2j_0 r^{1-\sigma} = 2j_0^\sigma,$$

  but decreasing for $\sigma \geqslant 1$, where the minimum is at

$$r = 1 \quad \Rightarrow \quad 2j_0 r^{1-\sigma} = 2j_0.$$

Overall, we find that the minimum of the right-hand side on $[0,1]$ is given by

$$\begin{cases} \frac{2j_0^\sigma}{2-\sigma}\left(\frac{2-\sigma}{\sigma}\right)^{\frac{\sigma}{2}}, & \sigma \in (0,1], \\ 2j_0, & \sigma \in [1,\infty). \end{cases}$$

Dividing through by $\sigma > 0$, gives us the sufficient condition

$$|\mathrm{p}_1| \leqslant \begin{cases} 2j_0^\sigma \sigma^{-1-\frac{\sigma}{2}}(2-\sigma)^{-1+\frac{\sigma}{2}}, & \sigma \in (0,1], \\ 2j_0 & \sigma \in [1,\infty). \end{cases}$$

□

We can generalise this result by using Theorem 2.24 instead.

Proposition 2.32. *Let* p_1 *satisfy*

$$\sigma\,|\mathrm{p}_1| \leqslant \begin{cases} \frac{1}{(1-\rho)^{1-\rho}\rho^\rho} C_\alpha^{-\rho}, & \rho \in \left(0,\frac{1}{2}\right], \\ 2C_\alpha^{-\frac{1}{2}}, & \rho \in \left[\frac{1}{2},\infty\right), \end{cases} \qquad \rho := \frac{\sigma}{2\alpha},$$

for some $\alpha > 0$. *Then* $\mathbb{E}_{\mathrm{p}} \geqslant 0$.

Proof. This time we use Theorem 2.16 to get the following sufficient condition.

$$\sigma\,|\mathrm{p}_1| \leqslant \begin{cases} r^{-\sigma} + \frac{1}{C_\alpha} r^{2\alpha-\sigma}, & r \in \left[0, C_\alpha^{\frac{1}{2\alpha}}\right], \\ \frac{2}{\sqrt{C_\alpha}} r^{\alpha-\sigma} & r \in \left[C_\alpha^{\frac{1}{2\alpha}}, 1\right], \end{cases} \qquad \text{a.e. } r \in [0,1].$$

Again, we minimise the right-hand side with respect to r.

- First consider $r \in \left[0, C_\alpha^{\frac{1}{2\alpha}}\right]$. Then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-\sigma} + \frac{1}{C_\alpha} r^{2\alpha-\sigma}\right) = -\sigma r^{-\sigma-1} + \frac{1}{C_\alpha}(2\alpha-\sigma) r^{2\alpha-\sigma-1} = r^{-\sigma-1}\left(\frac{1}{C_\alpha}(2\alpha-\sigma)r^{2\alpha} - \sigma\right).$$

  Thus, if $\sigma \leqslant \alpha$, we have a local minimum at

$$r = \left(\frac{\sigma C_\alpha}{2\alpha-\sigma}\right)^{\frac{1}{2\alpha}} \quad \Rightarrow \quad r^{-\sigma} + \frac{1}{C_\alpha} r^{2\alpha-\sigma} = \frac{2\alpha}{2\alpha-\sigma}\left(\frac{2\alpha-\sigma}{\sigma C_\alpha}\right)^{\frac{\sigma}{2\alpha}} = \frac{1}{(1-\rho)^{1-\rho}\rho^\rho} C_\alpha^{-\rho}.$$

  However, if $\sigma \geqslant \alpha$, $r \mapsto r^{-\sigma} + \frac{1}{\alpha} r^{2\alpha-\sigma}$ is decreasing on $\left[0, C_\alpha^{\frac{1}{2\alpha}}\right]$ and so the minimum occurs at

$$r = C_\alpha^{\frac{1}{2\alpha}} \quad \Rightarrow \quad r^{-\sigma} + \frac{1}{C_\alpha} r^{2\alpha-\sigma} = 2C_\alpha^{-\frac{\sigma}{2\alpha}} = 2C_\alpha^{-\rho}.$$

- Now we consider $r \in \left[C_\alpha^{\frac{1}{2\alpha}}, 1\right]$. For this interval to be non-empty, we require that $\alpha \geqslant \alpha^*$ so $C_\alpha \leqslant 1$. We then have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{2}{\sqrt{C_\alpha}} r^{\alpha-\sigma}\right) = \frac{2}{\sqrt{C_\alpha}}(\alpha-\sigma)r^{\alpha-\sigma-1}.$$

Hence, $r \mapsto \frac{2}{\sqrt{C_\alpha}} r^{\alpha-\sigma}$ is increasing for $\sigma \leqslant \alpha$ and so we have a minimum at

$$r = C_\alpha^{\frac{1}{2\alpha}} \quad \Rightarrow \quad \frac{2}{\sqrt{C_\alpha}} r^{\alpha-\sigma} = 2C_\alpha^{-\frac{\sigma}{2\alpha}} = 2C_\alpha^{-\rho},$$

but decreasing for $\sigma \geqslant \alpha$, where the minimum is at

$$r = 1 \quad \Rightarrow \quad \frac{2}{\sqrt{C_\alpha}} r^{\alpha-\sigma} = 2C_\alpha^{-\frac{1}{2}}.$$

Note that, since $\alpha \geqslant \alpha^*$, we have $C_\alpha \leqslant 1$ and so $2C_\alpha^{-\frac{1}{2}} \leqslant 2C_\alpha^{-\rho}$ for $\rho \geqslant \frac{1}{2}$.

Hence, we find that the minimum of the right-hand side on $[0,1]$ is given by

$$\begin{cases} \frac{1}{(1-\rho)^{1-\rho}\rho^\rho} C_\alpha^{-\rho}, & \rho \in \left(0, \frac{1}{2}\right], \\ 2C_\alpha^{-\frac{1}{2}}, & \rho \in \left[\frac{1}{2}, \infty\right). \end{cases}$$

□

It is then natural to wonder if we can pick an optimal α for a given σ. In other words, we wish to maximise

$$K_\sigma(\alpha) = \begin{cases} \frac{1}{(1-\rho)^{1-\rho}\rho^\rho} C_\alpha^{-\rho}, & \rho \in \left(0, \frac{1}{2}\right], \\ 2C_\alpha^{-\frac{1}{2}}, & \rho \in \left[\frac{1}{2}, \infty\right), \end{cases} \qquad \rho := \frac{\sigma}{2\alpha},$$

with respect to $\alpha > 0$. With some help form a computer algebra package, we can differentiate K to find that $K_\sigma'(\alpha) > 0$ for $\rho < \frac{1}{2}$ and $K_\sigma'(\alpha) < 0$ for $\rho > \frac{1}{2}$. Hence, we have a maximum at $\rho = \frac{1}{2}$ (i.e: $\alpha = \sigma$), where

$$K_\sigma(\sigma) = \frac{2}{\sqrt{C_\sigma}}.$$

Thus,

$$|\mathsf{p}_1| \leqslant \frac{2}{\sigma\sqrt{C_\sigma}} \quad \Rightarrow \quad \mathbb{E}_{\mathsf{p}} \geqslant 0. \tag{2.22}$$

For completeness, we shall also try using the critical version of the Hardy inequality as used in Theorem 2.28.

Proposition 2.33. *Let $k(\sigma)$ be the unique real root of the cubic polynomial*

$$p_\sigma(x) = 4\sigma x^3 - 12\sigma x^2 + 13\sigma x - 5\sigma + 2.$$

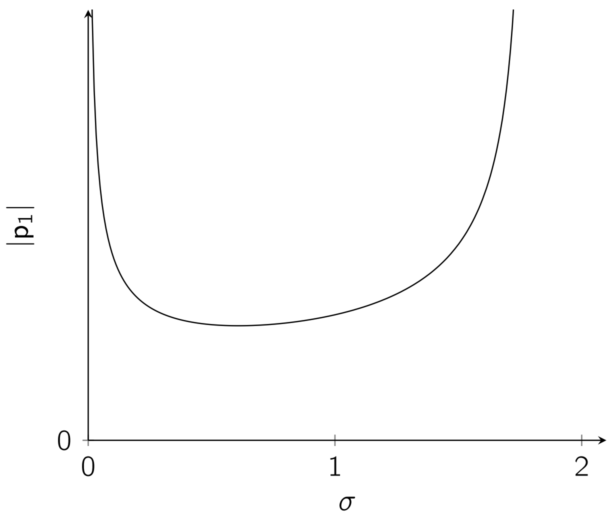

Figure 2.6: Plot of the optimal upper threshold for $|\mathsf{p}_1|$ using Proposition 2.32.

Then, if p_1 *satisfies*

$$|\mathsf{p}_1| \leqslant \begin{cases} \frac{1}{\sigma e^{k(\sigma)}}\left(1+\frac{1}{4(1-k(\sigma))^2}\right), & \sigma \in \left(0, \frac{2}{5}\right], \\ \frac{5}{4\sigma}, & \sigma \in \left[\frac{2}{5}, \infty\right), \end{cases} \tag{2.23}$$

we have $\mathbb{E}_{\mathsf{p}} \geqslant 0$*.*

Proof. We use Theorem 2.28 to get the sufficient condition

$$\sigma\,|\mathsf{p}_1| \leqslant r^{-\sigma} + \frac{1}{4r^{\sigma}(1-\log(r))^2},$$

and differentiate the right-hand side to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-\sigma} + \frac{1}{4r^{\sigma}(1-\log(r))^2}\right) = \frac{1}{4r^{\sigma+1}(1-\log(r))^3}p_{\sigma}(\log(r)).$$

Thus, if $\sigma \leqslant \frac{2}{5}$, we get a local minimum at $r = e^{k(\sigma)} \in (0,1]$, given by

$$r^{-\sigma} + \frac{1}{4r^{\sigma}(1-\log(r))^2} = e^{-k(\sigma)}\left(1+\frac{1}{4(1-k(\sigma))^2}\right).$$

However, if $\sigma \geqslant \frac{2}{5}$, then $r \mapsto r^{-\sigma} + \frac{1}{4r^{\sigma}(1-\log(r))^2}$ is decreasing on $[0,1]$ and so the minimum occurs at $r = 1$, where

$$r^{-\sigma} + \frac{1}{4r^{\sigma}(1-\log(r))^2} = \frac{5}{4}.$$

□

We note that, for sufficiently large $\sigma > 0$, we get a better sufficient condition (larger upper bound for $|\mathsf{p}_1|$) by using Proposition 2.32 with an optimally chosen α as opposed to Proposition 2.33 (see Figure 2.7).

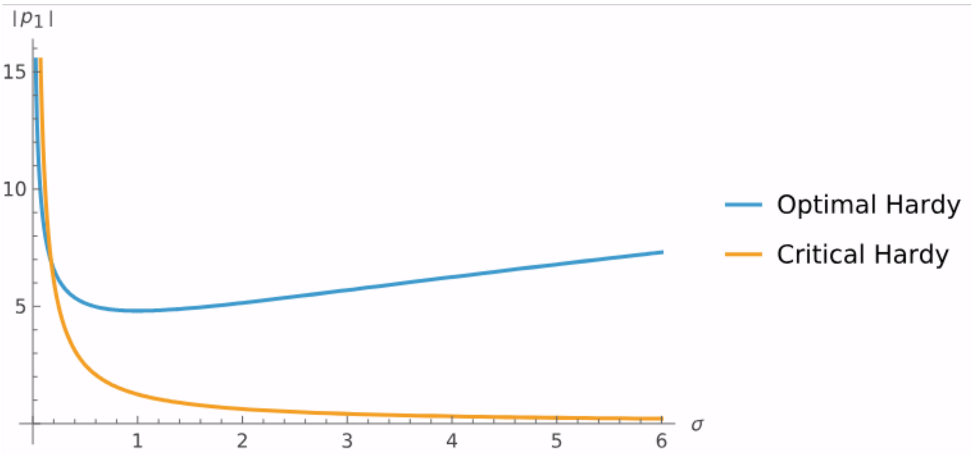

Figure 2.7: Plot comparing the upper threshold for $|\mathsf{p}_1|$ using the optimal version of the Hardy inequality (2.22) and the critical Hardy inequality (2.23) for varying $\sigma > 0$.

2.4.2. Logarithmic Pressure Functions

We now consider a logarithmic pressure function $\mathsf{p} : B \to \mathbb{R}$, given by

$$\mathsf{p}(r) = \mathsf{p}_0 + \mathsf{p}_* \log(r).$$

We note that, regardless of whether we use Theorem 2.16, Theorem 2.24 or Theorem 2.28, we get a sufficient condition of the form

$$|\mathsf{p}_*| \leqslant \rho(r),$$

where $\rho : [0, 1] \to \mathbb{R}$ is a continuous increasing function with $\rho(0) = 1$. This gives the following result.

Proposition 2.34. *If* $|\mathsf{p}_*| \leqslant 1$*, then* $\mathbb{E}_\mathsf{p} \geqslant 0$.

Logarithmic pressure functions also serve as an example in which we must use the sufficient conditions for radially symmetric functions as opposed to those that allow non-radially symmetric functions, which are not applicable to such low regularity pressure functions.

2.4.3. Linear Pressure Functions

We shall go over one example of a non-radially symmetric pressure, namely, a linear pressure function of the form

$$\mathsf{p}(x) = \lambda \cdot x, \qquad \lambda \in \mathbb{R}^2.$$

We can then try to find conditions on $\lambda \in \mathbb{R}^2$ to ensure $\mathbb{E}_\mathsf{p} \geqslant 0$.

Proposition 2.35. *If* $|\lambda| \leqslant j_0$*, then* $\mathbb{E}_\mathsf{p} \geqslant 0$.

Proof. Since $\nabla \mathbf{p} = \lambda$, we find that $\|\nabla \mathbf{p}\|_\infty = |\lambda|$. Then, using Proposition 2.18, we see that $\|\nabla \mathbf{p}\|_\infty = |\lambda| \leqslant j_0$ is sufficient for $\mathbb{E}_\mathbf{p} \geqslant 0$. □

This result can not be improved by using Proposition 2.26 instead of Proposition 2.18 since

$$\left\| |x|^{1-\alpha} \nabla \mathbf{p} \right\|_\infty = \begin{cases} |\lambda|, & \alpha \leqslant 1, \\ \infty, & \alpha > 1, \end{cases}$$

and $\alpha \mapsto \frac{1}{\sqrt{C_\alpha}}$ is decreasing on $(0,1]$. Hence $\alpha = 1$ gives the best sufficient condition using a Hardy inequality. We also don't get any improvement by Proposition 2.30 as $\| |x| (1 - \log(|x|)) \nabla \mathbf{p} \|_\infty = |\lambda|$, also occurring as $x \to \frac{\lambda}{|\lambda|}$, and so, by Proposition 2.30, we have

$$|\lambda| \leqslant \frac{1}{2} \quad \Rightarrow \quad \mathbb{E}_\mathbf{p} \geqslant 0,$$

and $\frac{1}{2} \leqslant j_0$.

2.5. Bounding the Dirichlet Energy in a Constrained Class

We now return to the task of minimising the Dirichlet energy $\mathbb{D}$ in the constrained class $\mathcal{A}_u$. We will show that the techniques used to bound the excess functional $\mathbb{E}_\mathbf{p}$ can also be applied directly to the Dirichlet energy and the constraint to derive lower bounds for $\mathbb{D}$ on $\mathcal{A}_u$. In this section we will focus on the example of u being the n-covering map.

$$u = u_n := \frac{r}{\sqrt{n}} e_r(n\theta) \qquad n \in \mathbb{N}.$$

This choice of $u_n \in H^1(B; \mathbb{R}^2)$ satisfies

$$\det \nabla u_n(x) = 1 \qquad \forall x \neq 0, \tag{2.24}$$

and

$$\mathbb{D}(u_n) = \frac{\pi}{2}\left(n + \frac{1}{n}\right).$$

Proposition 2.36. *Let $v \in \mathcal{A}_{u_n}$. Then*

$$\mathbb{D}(v) = \frac{\pi}{4}\int_0^1 r \left|V_0'(r)\right|^2 \,\mathrm{d}r + \frac{\pi}{2}\sum_{j>0}\int_0^1 r\left|V_j'(r)\right|^2 + \frac{j^2}{r}\left|V_j(r)\right|^2 \,\mathrm{d}r,$$

and

$$\sum_{j>0} j \det V_j(r) = r^2 \qquad a.e.\ r \in [0,1]. \tag{2.25}$$

Proof. The derivation of $\mathbb{D}(v)$ follows by using (2.3) similarly to the proof of Proposition 2.14. For the constraint we integrate (2.24) over $\theta \in [0, 2\pi)$ and Fourier decompose using (2.6). This gives us

$$\int_0^{2\pi} \det \nabla v \, \mathrm{d}x = \pi \sum_{j>0} \frac{j}{r} (\det V_j(r))'.$$

Hence, we find that

$$\sum_{j>0} j(\det V_j(r))' = 2r \quad \Rightarrow \quad \sum_{j>0} j \det V_j(r) = r^2,$$

for almost every $r \in [0, 1]$. Here we use the boundary condition $V_j(0) = 0$ for each $j > 0$ to do the integration. □

We again discard information contained in the higher order modes of the constraint to avoid mode mixing terms that would make calculations more cumbersome.

Lemma 2.37. *Let $\lambda \in W^{1,1}(B, \mathbb{R}^+)$ be radially symmetric. If we define*

$$D_j(\lambda; V) = \frac{\pi}{2} \int_0^1 r \left|V'(r)\right|^2 + \frac{j^2 - j\lambda(r)}{r} |V(r)|^2 \, \mathrm{d}r,$$

$$R(\lambda) = \pi \int_0^1 r\lambda(r) \, \mathrm{d}r,$$

then

$$\mathbb{D}(v) \geqslant \frac{1}{2} D_0(\lambda; V_0) + \sum_{j>0} D_j(\lambda; V_j) + R(\lambda) \qquad \forall v \in \mathcal{A}_{u_n}.$$

Proof. We first add and subtract $\frac{j\lambda(r)}{r} \left|V_j(r)\right|^2$ in the integrand and then make use of Hadamard's inequality and the constraint (2.25).

$$\begin{aligned}
\mathbb{D}(v) &= \frac{\pi}{4} \int_0^1 r \left|V_0'(r)\right|^2 \mathrm{d}r + \frac{\pi}{2} \sum_{j>0} \int_0^1 r \left|V_j'(r)\right|^2 + \frac{j^2 - j\lambda(r)}{r} \left|V_j(r)\right|^2 + \frac{j\lambda(r)}{r} \left|V_j(r)\right|^2 \mathrm{d}r \\
&\geqslant \frac{\pi}{4} \int_0^1 r \left|V_0'(r)\right|^2 \mathrm{d}r + \frac{\pi}{2} \sum_{j>0} \int_0^1 r \left|V_j'(r)\right|^2 + \frac{j^2 - j\lambda(r)}{r} \left|V_j(r)\right|^2 + \frac{2j\lambda(r)}{r} \det V_j(r) \, \mathrm{d}r \\
&= \frac{\pi}{4} \int_0^1 r \left|V_0'(r)\right|^2 \mathrm{d}r + \frac{\pi}{2} \sum_{j>0} \int_0^1 r \left|V_j'(r)\right|^2 + \frac{j^2 - j\lambda(r)}{r} \left|V_j(r)\right|^2 \mathrm{d}r + \pi \int_0^1 r\lambda(r) \, \mathrm{d}r.
\end{aligned}$$

□

Our goal is to show that u_n minimises $\mathbb{D}$ in $\mathcal{A}_{u_n}$ so we shall compare $D_j(V_j)$ with $D_j(U_j^n)$ for each $j \geqslant 0$. Here

$$U_j^n(r) = \delta_{j,n} \frac{r}{\sqrt{n}} I \quad \Rightarrow \quad D_j(\lambda; U_j^n) = 0 \qquad \forall j \neq n.$$

This motivates the restriction of $\lambda \leqslant \lambda_*$ where

$$\lambda_*(r) = \begin{cases} 1 + j_0^2 r^2, & r \in \left[0, \frac{1}{j_0}\right], \\ 2 j_0 r, & r \in \left[\frac{1}{j_0}, 1\right], \end{cases}$$

so $D_j(\lambda; V_j) \geqslant 0 = D_j(\lambda; U_j^n)$ for every $j \neq n$. The derivation of this inequality follows similarly to that in the proof of Theorem 2.16 since we now have $V_j(1) = U_j(1) = 0$ for $j \neq n$. We then have the implication

$$D_n(\lambda; V_n) + R(\lambda) \geqslant \frac{\pi}{2}\left(n + \frac{1}{n}\right) \quad \Rightarrow \quad \mathbb{D}(v) \geqslant \mathbb{D}(u_n).$$

With this in mind, we now seek to minimise $D_n(\lambda; V_n) + R(\lambda)$ subject to the constraint $\lambda \leqslant \lambda_*$.

Lemma 2.38. *Let $\lambda \leqslant \lambda^*$ be radially symmetric. If there exists a solution to the weak Euler-Lagrange equation for $D_n(\lambda; \cdot)$, then there is a minimiser of $D_n(\lambda; \cdot)$ in*

$$\mathcal{V}_n := \left\{ V \in W^{1,1}([0,1], \mathbb{R}^{2\times 2}) : \int_0^1 \frac{1}{r} |V(r)|^2 \, \mathrm{d}r < \infty, \int_0^1 r \left|V'(r)\right|^2 \mathrm{d}r < \infty, \ V(1) = \frac{1}{\sqrt{n}} I \right\}.$$

Proof. We start by defining a space of variations

$$\mathcal{V}_0 := \left\{ V \in W^{1,1}([0,1], \mathbb{R}^{2\times 2}) : \int_0^1 \frac{1}{r} |V(r)|^2 \, \mathrm{d}r < \infty, \int_0^1 r \left|V'(r)\right|^2 \mathrm{d}r < \infty, \ V(1) = 0 \right\},$$

and then perturb the energy to obtain

$$D_n(V) = D_n(V^* + \Phi) = D_n(V^*) + 2\langle V^*, \Phi \rangle_n + D_n(\Phi),$$

where the dependence on λ is implicit and $\Phi := V - V^* \in \mathcal{V}_0$ with $V^* \in \mathcal{V}_n$. The bilinear form $\langle \cdot, \cdot \rangle_n$ is defined by

$$\langle A, B \rangle_n := \frac{\pi}{2} \int_0^1 r A'(r) \cdot B'(r) + \frac{n^2 - n\lambda(r)}{r} A(r) \cdot B(r) \, \mathrm{d}r.$$

Now suppose V^* solves the weak Euler-Lagrange equation for D_n. Then, by approximating $\Phi \in \mathcal{V}_0$ by smooth functions, we observe that

$$\langle V^*, \Phi \rangle_n = 0 \qquad \forall \Phi \in \mathcal{V}_0.$$

Since $\lambda \leqslant \lambda^*$, we know that $D_n(\Phi) \geqslant 0$ for all $\Phi \in \mathcal{V}_0$ (see proof of Theorem 2.16 for details). Hence, we have

$$D_n(V) = D_n(V^* + \Phi) = D_n(V^*) + 2\langle V^*, \Phi \rangle_n + D_n(\Phi) \geqslant D_n(V^*) \qquad \forall V \in \mathcal{V}_n,$$

and so V^* minimises D_n in $\mathcal{V}_n$. $\square$

It now remains to solve the Euler-Lagrange equation for D_n. The Euler-Lagrange equation is given by

$$r^2V''(r) + rV'(r) - (n^2 - n\lambda(r))V(r) = 0 \qquad V(0) = 0,\ V(1) = \frac{1}{\sqrt{n}}I. \tag{2.26}$$

We must also consider the optimal λ to pick. To do this, we shall fix $V^* \in \mathcal{V}_n$ and consider the Euler-Lagrange equation of $D_n(\lambda; V^*) + R(\lambda)$ with respect to λ. We then obtain

$$\frac{n}{r}|V^*(r)|^2 - 2r = 0. \tag{2.27}$$

We would like to show that $V^* = U_n^n$ is the minimiser. Indeed, we see that $V^* = U_n^n$ satisfies (2.27) and solves (2.26) if

$$\lambda(r) = n - \frac{1}{n} \qquad \text{a.e. } r \in [0,1].$$

This introduces the main limitation of this technique as

$$n - \frac{1}{n} > 1 = \lambda^*(0) \geqslant \lambda(0) \qquad \forall n > 1.$$

Thus, with the exception of the identity map ($n = 1$), we cannot use this method to show that u_n minimises $\mathbb{D}$ in $\mathcal{A}_{u_n}$. However, we can derive lower bounds for $\mathbb{D}$ in $\mathcal{A}_{u_n}$ that come close to the desired minimisation result for small n. With this in mind, we now take

$$\lambda(r) = \lambda_n(r) := \min\left\{\lambda^*(r), n - \frac{1}{n}\right\} \qquad r \in [0,1],$$

so $\lambda \leqslant \lambda^*$.

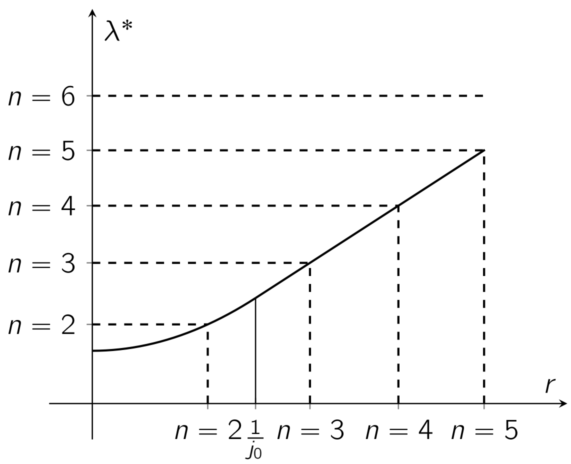

Figure 2.8: The threshold function λ^* with dashed lines indicating where $\lambda^* = n - \frac{1}{n}$ for $n = 2, 3, 4, 5, 6$.

Theorem 2.39. *Let V_n solve the Euler-Lagrange equation for $D_n(\lambda_n; \cdot)$. Then*

$$\mathbb{D}(v) \geqslant D_n(\lambda_n; V_n) + R(\lambda_n) \qquad \forall v \in \mathcal{A}_{u_n}.$$

Proof. This follows immediately from Lemmas 2.37 and 2.38. □

As an example, we shall apply this theorem in the case of $n = 2$. Here the Euler-Lagrange equation for V_2 is split into two parts, namely

$$\begin{cases} r^2 V_2''(r) + rV_2'(r) - 2(1 - j_0^2 r^2)V_2(r) = 0, & r \in \left[0, \frac{1}{j_0\sqrt{2}}\right], \\ r^2 V_2''(r) + rV_2'(r) - V_2(r) = 0, & r \in \left[\frac{1}{j_0\sqrt{2}}, 1\right], \end{cases}$$

with boundary conditions $V_2(0) = 0$ and $V_2(1) = \frac{1}{\sqrt{2}} I$. Under a change of variables $s = \sqrt{2} j_0 r$, the first part of the Euler-Lagrange equation becomes a Bessel equation of order $\sqrt{2}$, whereas the second part is a Cauchy-Euler equation. Hence, we can find a C^1 solution given by $V_2(r) = \nu_2(r) I$, where

$$\nu_2(r) = \begin{cases} A J_{\sqrt{2}}\left(\sqrt{2} j_0 r\right), & r \in \left[0, \frac{1}{j_0\sqrt{2}}\right], \\ B\left(r^{-1} - r\right) + \frac{r}{\sqrt{2}}, & r \in \left[\frac{1}{j_0\sqrt{2}}, 1\right], \end{cases}$$

and

$$A \approx 0.707063, \qquad B \approx -0.00561494.$$

These constants are obtained by numerically solving the continuity equation for ν_2 and ν_2' at $r = \frac{1}{j_0\sqrt{2}}$. We then calculate the lower bound in Theorem 2.39 to be

$$D_2(\lambda_2; V_2) + R(\lambda_2) \approx 3.91799 = \rho \mathbb{D}(u_2), \qquad \rho \approx 0.997707,$$

bringing us within 0.223% of the desired minimisation result.

3

Unbounded Pressure Functions

We have now established sufficient conditions for non-negativity of the excess functional that apply to bounded or differentiable pressure functions. However, the excess functional is defined for any $\mathsf{p} \in \mathrm{BMO}$ and does not need to be bounded or differentiable. In this chapter, we will explore new methods for deriving sufficient conditions for $\mathbb{E}_{\mathsf{p}} \geqslant 0$ that can be applied to arbitrary $\mathsf{p} \in \mathrm{BMO}$ and do not rely on boundedness or differentiability. The key idea is to use the fact that $\det \nabla\varphi \in \mathcal{H}^1$, a space that is dual to BMO [29, 38], for any $\varphi \in H_0^1$. Furthermore, there exists a constant $C_\delta > 0$ such that

$$\|\det \nabla\varphi\|_{\mathcal{H}^1} \leqslant \frac{1}{2} C_\delta \|\nabla\varphi\|_2^2 . \tag{3.1}$$

We can then use duality to bound the excess as

$$\mathbb{E}_{\mathsf{p}}(\varphi) \geqslant \frac{1}{2} \|\nabla\varphi\|_2^2 - C_* [\mathsf{p}]_{\mathrm{BMO}} \cdot \|\det \nabla\varphi\|_{\mathcal{H}^1} ,$$

If we pick the norms for BMO and $\mathcal{H}^1$ according to the duality relation, then the constant is just $C_* = 1$. However, we will allow for the choice of any equivalent norms, and thus pick up a constant $C_* > 0$ that depends only on the domain. Then, using (3.1), we can bound the excess like so:

$$\mathbb{E}_{\mathsf{p}}(\varphi) \geqslant \frac{1}{2} \|\nabla\varphi\|_2^2 - \frac{1}{2} C_* C_\delta [\mathsf{p}]_{\mathrm{BMO}} \cdot \|\nabla\varphi\|_2^2 = \frac{1}{2}(1 - C_* C_\delta [\mathsf{p}]_{\mathrm{BMO}}) \|\nabla\varphi\|_2^2 .$$

This gives a sufficient condition for the non-negativity of the excess functional $\mathbb{E}_{\mathsf{p}}$ in terms of the seminorm $[\mathsf{p}]_{\mathrm{BMO}}$.

Theorem 3.1. *Let $\mathsf{p} \in \mathrm{BMO}(B)$ be such that $[\mathsf{p}]_{\mathrm{BMO}} \leqslant \frac{1}{C_* C_\delta}$. Then*

$$\mathbb{E}_{\mathsf{p}}(\varphi) \geqslant 0 \qquad \forall \varphi \in H_0^1(B; \mathbb{R}^2).$$

It remains to compute the constants C_* and C_δ to make this result explicit.

3.1. Compensated Compactness

In this section, we will calculate an explicit value for the constant C_δ in (3.1). We shall start by defining the Hardy space $\mathcal{H}^1$ using the so-called grand maximal function along with some key properties. We will then move on to the calculation of the constant C_δ which arises from a div-curl lemma.

3.1.1. Maximal Characterisation of Hardy Spaces

We shall start by giving a standard definition for the Hardy space $\mathcal{H}^1$ and an appropriate norm [29, 38]. We first define a class of test functions $\mathcal{T} \subset C^\infty$ by

$$\mathcal{T} := \left\{\phi \in C^\infty(\mathbb{R}^2) : \operatorname{supp}\phi \subset B \text{ and } \|\nabla\phi\|_\infty \leqslant 1\right\}.$$

For each $\phi \in \mathcal{T}$ and $t > 0$, we define the corresponding function $\phi_t \in C^\infty(\mathbb{R}^2)$ by

$$\phi_t(x) = \frac{1}{t^2}\phi\left(\frac{x}{t}\right).$$

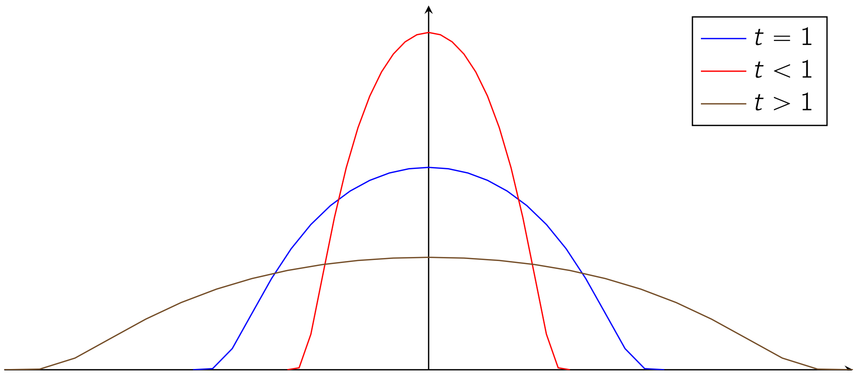

Figure 3.1: Plots of ϕ_t for varying $t > 0$ and fixed $\phi \in \mathcal{T}$ in one dimension.

Observe that $\operatorname{supp}\phi_t \subset B_t$ and

$$\|\nabla\phi_t\|_\infty = t^{-3}\|\nabla\phi\|_\infty \leqslant t^{-3}.$$

This allows us to define the grand maximal function which can be used to characterise Hardy spaces.

Definition 3.2. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^2)$. The grand maximal function of f is $f^* : \mathbb{R}^2 \to [0, \infty)$, defined by*

$$f^*(x) = \sup_{\phi\in\mathcal{T}}\sup_{t>0}|(f * \phi_t)(x)|.$$

We can now define the Hardy space $\mathcal{H}^1$ in terms of the grand maximal function.

Definition 3.3. *We define the space $\mathcal{H}^1(\mathbb{R}^2)$ to be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ such that $f^* \in L^1$. Furthermore, we define a norm on $\mathcal{H}^1$ by*

$$\|f\|_{\mathcal{H}^1} = \|f^*\|_1 .$$

The Hardy space $\mathcal{H}^1$ has a number of key properties [29] that are frequently used in harmonic analysis. One such property is its relation to L^1.

Lemma 3.4. *[29, 4.2.7] $\mathcal{H}^1(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$ with*

$$\|f\|_1 \leqslant \|f\|_{\mathcal{H}^1} .$$

Next, we note that the functions in $\mathcal{H}^1$ have vanishing mean.

Lemma 3.5. *[29, 4.2.8] Let $f \in \mathcal{H}^1(\mathbb{R}^2)$. Then*

$$\int_{\mathbb{R}^2} f(x)\, dx = 0.$$

Conversely, it is worth noting that there exist functions in L^1 with vanishing mean that are not in $\mathcal{H}^1$. We also have a relationship between $\mathcal{H}^1$ and the L^p spaces.

Lemma 3.6. *[29, 4.2.9] Let $f \in L^p(\mathbb{R}^2)$ for $p \in (1, \infty)$ be compactly supported with vanishing mean. Then $f \in \mathcal{H}^1(\mathbb{R}^2)$.*

Again, if we consider the converse, there are functions in $\mathcal{H}^1$ which fail to be L^p for any $p > 1$.

3.1.2. The Div-Curl Lemma

It is a common exercise in functional analysis to determine the regularity of a product of two functions $f \cdot g$, given the regularity of f and g. This is often done via Hölder's inequality.

$$\|f \cdot g\|_1 \leqslant \|f\|_p \|g\|_q , \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.2}$$

Recall that L^1 sits strictly inside $\mathcal{H}^1$ and so replacing the L^1 norm with the $\mathcal{H}^1$ norm in 3.2 would give improved regularity. It turns out that this indeed holds under certain assumptions.

Theorem 3.7. *[29, 4.2.12] Let $f \in L^p(\mathbb{R}^2, \mathbb{R}^2)$ be weakly divergence-free and $g \in L^q(\mathbb{R}^2, \mathbb{R}^2)$ be weakly curl-free for $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $f \cdot g \in \mathcal{H}^1(\mathbb{R}^2)$ with*

$$\|f \cdot g\|_{\mathcal{H}^1} \leqslant C_\delta^{p,q} \|f\|_p \|g\|_q ,$$

for some constant $C_\delta^{p,q} > 0$ depending on p and q.

To prove this theorem, we will need to make use of the Hardy-Littlewood maximal inequality [29].

Definition 3.8. *Let $f \in L^1(\mathbb{R}^2)$. The Hardy-Littlewood maximal function $Mf : \mathbb{R}^2 \to [0, \infty)$ is given by*

$$(Mf)(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)| \, \mathrm{d}y.$$

Lemma 3.9. *Let $f \in L^p(\mathbb{R}^2)$ for $p \in (1, \infty)$. Then*

$$\|Mf\|_p \leqslant C_{HL}^p \|f\|_p ,$$

with a bound for the constant given by $C_{HL}^p = 2\left(\frac{25p}{p-1}\right)^{\frac{1}{p}}$.

We now prove Theorem 3.7 following the standard method [16] but ensuring that we obtain an explicit bound for the constant.

Proof of Theorem 3.7. First note that since g is weakly curl-free, we can write it as $g = \nabla\xi$ for some $\xi \in W^{1,q}(\mathbb{R}^2)$. We start by considering an arbitrary $\phi \in \mathcal{T}$ and then let $x \in \mathbb{R}^2$ and $t > 0$. For convenience, we shall make use of the notation

$$\langle\xi\rangle_t^x := \fint_{B_t(x)} \xi(y) \, \mathrm{d}y.$$

We now consider the convolution $\phi_t * (f \cdot g) = \phi_t * (f \cdot \nabla\xi)$. Making use of the fact that f is weakly divergence-free, we can integrate by parts to get

$$\begin{aligned}
|(\phi_t * (f \cdot g))(x)| &= \left| \int_{B_t(x)} \frac{1}{t^2}\phi\left(\frac{x-y}{t}\right) (f(y) \cdot \nabla(\xi(y) - \langle\xi\rangle_t^x)) \, \mathrm{d}y \right| \\
&= \left| \int_{B_t(x)} \frac{1}{t^2}\phi\left(\frac{x-y}{t}\right) \operatorname{div}(f(y)(\xi(y) - \langle\xi\rangle_t^x)) \, \mathrm{d}y \right| \\
&= \left| \int_{B_t(x)} \frac{1}{t^3}\nabla\phi\left(\frac{x-y}{t}\right) \cdot (f(y)(\xi(y) - \langle\xi\rangle_t^x)) \, \mathrm{d}y \right| \\
&\leqslant \frac{\|\nabla\phi\|_\infty}{t} \left| \int_{B_t(x)} \frac{1}{t^2} (f(y)(\xi(y) - \langle\xi\rangle_t^x)) \, \mathrm{d}y \right| \\
&\leqslant \frac{1}{t} \left| \int_{B_t(x)} \frac{1}{t^2} (f(y)(\xi(y) - \langle\xi\rangle_t^x)) \, \mathrm{d}y \right|.
\end{aligned}$$

Now let $\alpha \in (1, p)$ and pick α' such that $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Then we can apply Hölder's inequality to get

$$\begin{aligned}
|(\phi_t * (f \cdot g))(x)| &\leqslant \frac{1}{t} \left(\fint_{B_t(x)} |f(y)|^\alpha \, \mathrm{d}y \right)^{1/\alpha} \left(\fint_{B_t(x)} |\xi(y) - \langle\xi\rangle_t^x|^{\alpha'} \, \mathrm{d}y \right)^{1/\alpha'} \frac{|B_t(x)|}{t^2} \\
&= \frac{\pi}{t} \left(\fint_{B_t(x)} |f(y)|^\alpha \, \mathrm{d}y \right)^{1/\alpha} \left(\fint_{B_t(x)} |\xi(y) - \langle\xi\rangle_t^x|^{\alpha'} \, \mathrm{d}y \right)^{1/\alpha'},
\end{aligned}$$

using the fact that $|B_t(x)| = \pi t^2$. Now let $\beta \in (1,q)$ be such that $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{3}{2}$. Since $\alpha > 1$, we know that $\beta < 2$, and so we can apply the Gagliardo–Nirenberg–Sobolev inequality [5, 32]

$$\left(\int_{B_t(x)} |\xi(y) - \langle \xi \rangle_t^x|^{\alpha'} \, \mathrm{d}y\right)^{1/\alpha'} \leqslant C_{\mathrm{GNS}}^{\beta} \left(\int_{B_t(x)} |\nabla \xi(y)|^{\beta} \, \mathrm{d}y\right)^{1/\beta},$$

because

$$\frac{1}{\alpha'} = 1 - \frac{1}{\alpha} = 1 - \left(\frac{3}{2} - \frac{1}{\beta}\right) = \frac{1}{\beta} - \frac{1}{2}.$$

The optimal constant C_{GNS}^{β} for $\beta < 2$ has been discussed in the literature [5]. We then have that

$$\begin{aligned}
|(\phi_t * (f \cdot g))(x)| &\leqslant \frac{\pi}{t} |B_t(x)|^{\frac{1}{\beta}-\frac{1}{\alpha'}} C_{\mathrm{GNS}}^{\beta} \left(\fint_{B_t(x)} |f(y)|^{\alpha} \, \mathrm{d}y\right)^{1/\alpha} \left(\fint_{B_t(x)} |g(y)|^{\beta} \, \mathrm{d}y\right)^{1/\beta} \\
&= \pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left(\fint_{B_t(x)} |f(y)|^{\alpha} \, \mathrm{d}y\right)^{1/\alpha} \left(\fint_{B_t(x)} |g(y)|^{\beta} \, \mathrm{d}y\right)^{1/\beta}.
\end{aligned}$$

Taking the supremum over all $\phi \in \mathcal{T}$ and $t > 0$, we have

$$(f \cdot g)^*(x) \leqslant \pi^{3/2} C_{\mathrm{GNS}}^{\beta} M(|f|^{\alpha})(x)^{1/\alpha} M(|g|^{\beta})(x)^{1/\beta}.$$

Now we integrate and apply Hölder's inequality, using $\frac{1}{p}+\frac{1}{q}=1$.

$$\begin{aligned}
\|f \cdot g\|_{\mathcal{H}^1} &\leqslant \pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left(\int_{\mathbb{R}^2} M(|f|^{\alpha})(x)^{p/\alpha} \, \mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}^2} M(|g|^{\beta})(x)^{q/\beta} \, \mathrm{d}x\right)^{1/q} \\
&= \pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left\| M\left(|f|^{\alpha}\right) \right\|_{p/\alpha}^{1/\alpha} \cdot \left\| M\left(|g|^{\beta}\right) \right\|_{q/\beta}^{1/\beta}.
\end{aligned}$$

Then we can make use of the Hardy-Littlewood maximal inequality (Lemma 3.9) to get the estimate

$$\begin{aligned}
\|f \cdot g\|_{\mathcal{H}^1} &\leqslant \pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left(C_{\mathrm{HL}}^{p/\alpha} \left\||f|^{\alpha}\right\|_{p/\alpha}\right)^{1/\alpha} \left(C_{\mathrm{HL}}^{q/\beta} \left\||g|^{\beta}\right\|_{q/\beta}\right)^{1/\beta} \\
&= \pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left(C_{\mathrm{HL}}^{p/\alpha}\right)^{1/\alpha} \left(C_{\mathrm{HL}}^{q/\beta}\right)^{1/\beta} \|f\|_p \|g\|_q .
\end{aligned}$$

The required constant is then given by

$$C_{\delta}^{p,q} := \inf_{(\alpha,\beta)\in\mathcal{P}} \pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left(C_{\mathrm{HL}}^{p/\alpha}\right)^{1/\alpha} \left(C_{\mathrm{HL}}^{q/\beta}\right)^{1/\beta},$$

with $\mathcal{P} = \left\{(\alpha,\beta) \in (1,p) \times (1,q) : \frac{1}{\alpha}+\frac{1}{\beta}=\frac{3}{2}\right\}$. □

We can immediately apply this to the Jacobian so that we can bound $\mathbb{E}_{\mathsf{p}}$.

Corollary 3.10. *Let $\varphi \in H^1(B;\mathbb{R}^2)$. Then $\det \nabla\varphi \in \mathcal{H}^1$ with*

$$\|\det \nabla \varphi\|_{\mathcal{H}^1} \leqslant \frac{1}{2} C_{\delta} \|\nabla \varphi\|_2^2 .$$

Proof. We write $\det \nabla\varphi = \frac{1}{2}\nabla\varphi \cdot \operatorname{cof} \nabla\varphi$ and observe that $\nabla\varphi \in W^{1,2}(B;\mathbb{R}^2)$ is weakly curl-free and $\operatorname{cof} \nabla\varphi \in W^{1,2}(B;\mathbb{R}^2)$ is weakly divergence-free. Hence,

$$\|\det \nabla\varphi\|_{\mathcal{H}^1} \leqslant \frac{1}{2} C_\delta^{2,2} \|\nabla\varphi\|_2 \cdot \|\operatorname{cof} \nabla\varphi\|_2 = \frac{1}{2} C_\delta \|\nabla\varphi\|_2^2 ,$$

with $C_\delta := C_\delta^{2,2}$. □

By using numerical optimisation techniques, we can find that the infimum for C_δ is given by

$$C_\delta \approx 492.451$$

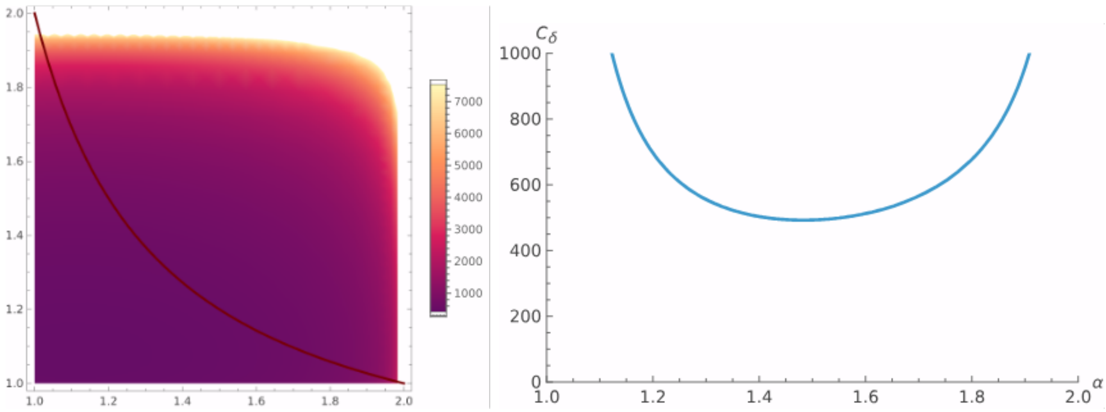

Figure 3.2: Left: Plot of $\pi^{3/2} C_{\mathrm{GNS}}^{\beta} \left(C_{\mathrm{HL}}^{2/\alpha}\right)^{1/\alpha} \left(C_{\mathrm{HL}}^{2/\beta}\right)^{1/\beta}$ for varying $\alpha, \beta \in (1,2)$ with the curve $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{2}$ highlighted. Right: A plot of the same function along said curve, parametrised by $\alpha \in (1,2)$.

3.2. Hardy-BMO Duality

The other key ingredient in the proof of Theorem 3.1 is the duality between BMO and $\mathcal{H}^1$, which will allow us to bound the Jacobian term in the excess functional using a Cauchy-Schwarz-like inequality. In this section, we shall introduce functions of bounded mean oscillation and their properties, along with atoms, in the context of Hardy spaces. We will then go over some fundamental results in harmonic analysis, improving the existing proofs by calculating explicit constants for several bounds. Finally, we will bring all of this together to derive an inequality for the product of BMO and $\mathcal{H}^1$ functions, providing a value for the constant C_*.

3.2.1. Bounded Mean Oscillation and Atoms

We shall start by defining the sharp maximal function, which will be used to characterise functions of bounded mean oscillation.

Definition 3.11. *Let $f \in L^1_{loc}(\mathbb{R}^2)$. The sharp maximal function $f^\sharp : \mathbb{R}^2 \to \mathbb{R}^2$ is given by*

$$f^\sharp(x) = \sup_{B \ni x} \fint_B |f(y) - f_B| \, \mathrm{d}y,$$

where the supremum is taken over all balls B containing x.

These sharp maximal functions are used to define the seminorm for the space of BMO functions.

Definition 3.12. *We define the space* $\mathrm{BMO}(\mathbb{R}^2)$ *to be the set of all $f \in L^1_{loc}(\mathbb{R}^2)$ such that $f^\sharp \in L^\infty$. Furthermore, we define a seminorm in* BMO *by*

$$[f]_{\mathrm{BMO}} = \left\| f^\sharp \right\|_\infty .$$

We note that $[\cdot]_{\mathrm{BMO}}$ is a seminorm but not a norm as $[c]_{\mathrm{BMO}} = 0$ for all constants $c \in \mathbb{R}$. Despite this, many authors will refer to $[\cdot]_{\mathrm{BMO}}$ as a norm by working in a quotient space where functions that differ by a constant are equivalent. For simplicity, we shall not use $[\cdot]_{\mathrm{BMO}}$ as a norm. However, we can still define a norm on BMO, such as

$$\|f\|_{\mathrm{BMO}} = \|f\|_1 + [f]_{\mathrm{BMO}} .$$

We now revisit Hardy spaces to introduce the concept of *atoms*, which will play a key role in showing the duality between $\mathcal{H}^1$ and BMO.

Definition 3.13. *We say $a \in L^1(\mathbb{R}^2)$ is a $\mathcal{H}^1$-atom if all the following hold:*

$$\operatorname{supp} a \subset B,$$
$$\|a\|_\infty \leqslant \frac{1}{|B|},$$
$$\int_{\mathbb{R}^2} a(x) \, \mathrm{d}x = 0,$$

for some ball $B \subset \mathbb{R}^2$.

We first note that $\mathcal{H}^1$-atoms are themselves in $\mathcal{H}^1$.

Lemma 3.14. *[29, 4.3.2] There exists a constant $A > 0$ such that $\|a\|_{\mathcal{H}^1} \leqslant A$ for all $\mathcal{H}^1$-atoms a.*

We can also use $\mathcal{H}^1$-atoms to build functions in $\mathcal{H}^1$.

Lemma 3.15. *[29, 4.3.3] Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{H}^1$-atoms and $\lambda \in \ell^1$ be a sequence of real numbers. Then $f = \sum_{k\in\mathbb{N}} \lambda_k a_k$ is in $\mathcal{H}^1$. Furthermore, there exists a constant $c > 0$, independent of f, such that*

$$\|f\|_{\mathcal{H}^1} \leqslant c \|\lambda\|_1 .$$

We will also show that the converse of this result holds, that is, that any function in $\mathcal{H}^1$ can be decomposed into $\mathcal{H}^1$ atoms, with a similar bound in place. However, before doing this, we will need to introduce some additional tools from harmonic analysis.

3.2.2. Covering Lemma and Calderon-Zygmund Decomposition

In this section, we will derive explicit bounds for constants in several results in harmonic analysis. Before we start, we will first prove a variation of the triangle inequality for later use.

Lemma 3.16. *Let A, B, C be subsets of $\mathbb{R}^2$ with A and B compact and C closed. Then*

$$d(A, C) \leqslant d(A, B) + \operatorname{diam} B + d(B, C).$$

In particular, if A and B have non-empty intersection, then

$$d(A, C) \leqslant \operatorname{diam} B + d(B, C).$$

Proof. We first need to show that there exists $y \in B$ and $c \in C$ such that $d(y, c) = d(B, C)$. We show this by considering infimising sequences $y_n \in B$ and $c_n \in C$ such that

$$d(y_n, c_n) \to d(B, C).$$

Since B is compact, we can take a convergent subsequence $y_{n_k} \to y$ and so we have

$$\lim_{k\to\infty} d\left(y_{n_k}, c_{n_k}\right) = \lim_{k\to\infty} d\left(y, c_{n_k}\right) = d(B, C),$$

using the continuity of d in each argument. Thus, c_{n_k} is a bounded sequence in a closed subset C of $\mathbb{R}^2$, so it has a convergent subsequence $c_{n_{k_j}} \to c$. We then use the continuity of d again to establish

$$d\left(y_{n_{k_j}}, c_{n_{k_j}}\right) \to d(y, c) = d(B, C)$$

Similarly, we can pick $a \in A$ and $x \in B$ such that $d(a, x) = d(A, B)$. We then define

$$\ell := d(a, c) \geqslant d(A, C),$$
$$\delta := d(x, y) \leqslant \operatorname{diam} B.$$

Then by the triangle inequality, we have

$$d(A, C) \leqslant d(a, c) \leqslant d(a, x) + d(x, y) + d(y, c) \leqslant d(A, B) + \operatorname{diam} B + d(B, C).$$

The second inequality follows from the fact that $d(A, B) = 0$ when $A \cap B \neq \varnothing$. □

Now we establish a covering lemma, often referred to as the Whitney decomposition, which will later form the basis for the proof of a Calderon-Zygmund decomposition. This lemma has previously been proven by Lamm [29] and Stein [38] but without a bound on the scale factor β.

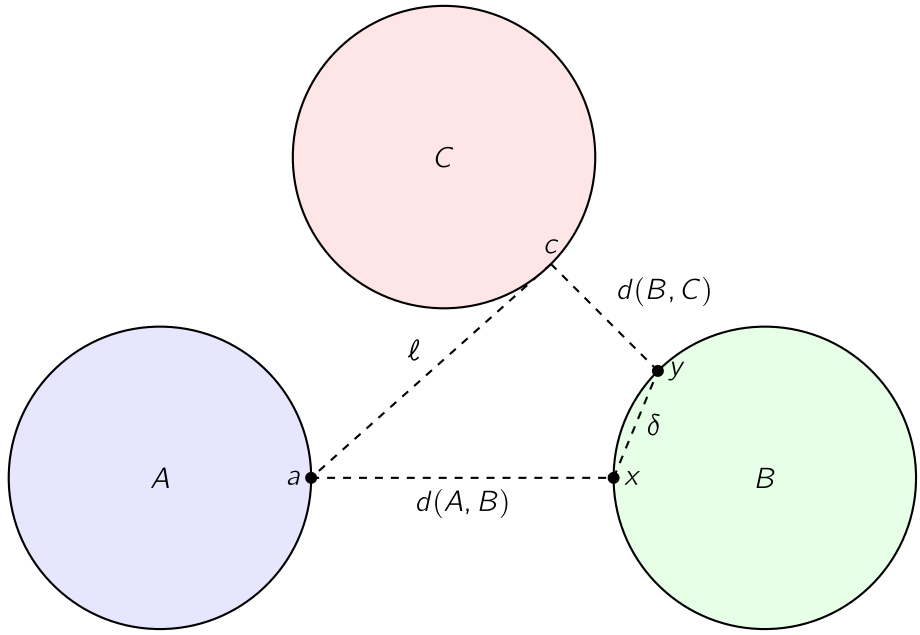

Figure 3.3: Diagram depicting the triangle inequality for subsets.

Lemma 3.17. *Suppose $F \subset \mathbb{R}^2$ is closed and non-empty, and let $\Omega = \mathbb{R}^2 \backslash F$. There exists a partition $\{Q_k\}_{k\in\mathbb{N}}$ of Ω into cubes Q_k satisfying*

$$\operatorname{diam} Q_k \leqslant d(Q_k, F) \leqslant 4 \operatorname{diam} Q_k \tag{3.3}$$

Moreover, if Q_k^* denotes the cube having the same centre as Q_k but with side lengths scaled by a factor of $\beta \in \left(1, \frac{4}{3}\right)$, then the collection $\{Q_k^*\}_{k\in\mathbb{N}}$ has the bounded intersection property and satisfies $\bigcup_{k\in\mathbb{N}} Q_k^* = \bigcup_{k\in\mathbb{N}} Q_k$.

Proof. For the construction of $\{Q_k\}_{k\in\mathbb{N}}$, refer to the proof of Lemma 4.3.4 given by Lamm [29]. As part of this proof, it is shown that neighbouring squares cannot be different in scale by more than a factor of 4. We will use this observation to construct the densest possible arrangement of squares and show that it is still possible to construct a rescaling of these squares with the bounded intersection property.

Now consider an arbitrary square $Q \in \{Q_k\}_{k\in\mathbb{N}}$. We then perform the following iterative procedure:

1. Around the outside of Q, form a layer of $n_1 = 20$ squares, each having a side length $\frac{1}{4}$ of the side length of Q.

2. Around the outside of the composite square, form a layer of $n_2 = 100$ squares, each having a side length $\frac{1}{4^2}$ of the side length of Q.

3. Repeat this process ad infinitum.

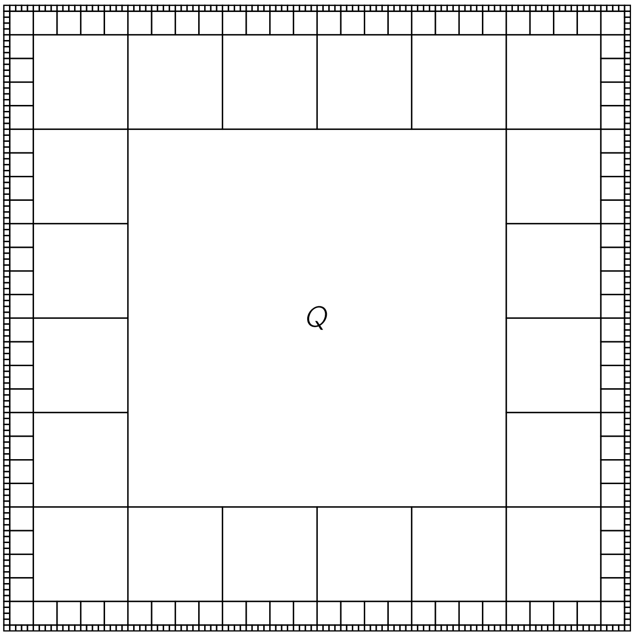

Figure 3.4: Diagram showing the first 3 layers formed around Q.

We calculate that the number of squares n_L in the L-th layer is given by

$$n_{L+1} = 4^2 \left(\frac{n_L - 4}{4} + 2 \right) + 4, \qquad n_0 = 0,$$

or, more explicitly,

$$n_L = \frac{20}{3} \left(4^L - 1 \right), \qquad L \in \mathbb{N}_0.$$

Hence, the total number of squares across all layers from 1 to $L \in \mathbb{N}$ is given by

$$N_L = \sum_{j=1}^{L} n_j = \frac{20}{9} \left(4^{L+1} - 3L - 4 \right).$$

We then pick β such that scaling Q by a factor of β covers all layers up to $L-1$ and a proportion $\epsilon \in (0,1)$ of the squares in layer L but also such that rescaling the squares in layer $L+1$ (or beyond) does not intersect with the rescaled Q. In other words, we take

$$\beta = \sum_{j=0}^{L-1} \frac{1}{4^j} + \frac{\epsilon}{4^L},$$

but must have

$$\beta \leqslant \sum_{j=0}^{L} \frac{1}{4^j} - \frac{\beta}{4^{L+1}}.$$

The maximum ϵ for which this holds is

$$\epsilon = \epsilon_L := \frac{4}{3} \cdot \frac{2^{2L+1} + 1}{2^{2L+2} + 1},$$

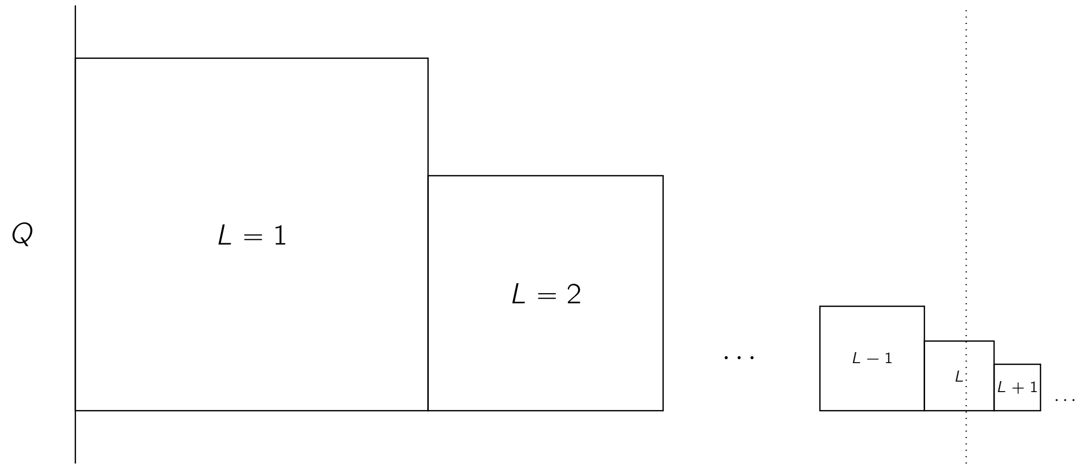

Figure 3.5: Diagram (not to scale) depicting the right edge of Q along with a square from each layer from 1 to $L+1$. The dotted line indicates the splitting of the square in layer L with proportions ϵ and $1-\epsilon$. The scaling β and the proportion ϵ are chosen such that the right edge of Q and the left edge of the square in layer $L+1$ meet along the dotted line after they are both scaled by a factor of β.

and the corresponding scaling factor is

$$\beta = \beta_L := \frac{4}{3} \cdot \frac{4^{L+1}-1}{4^{L+1}+1}.$$

So far, we have shown that for each $L \in \mathbb{N}$, there exists a scale factor β_L, such that the set $\{Q_k^*\}_{k\in\mathbb{N}}$, formed by rescaling the squares in $\{Q_k\}_{k\in\mathbb{N}}$ by β_L, has the bounded intersection property (each square intersects no more than N_{L+1} other squares). It then follows that the same result holds, provided that we take $\beta < \sup_{L\in\mathbb{N}} \beta_L = \frac{4}{3}$.
Finally, we note that the composite square formed is $\frac{4}{3}Q$ which sits inside Ω since $d(Q,F) \geqslant \operatorname{diam}(Q)$. Using the same reasoning, we know that $\bigcup_{k\in\mathbb{N}} Q_k^* \subset \Omega = \bigcup_{k\in\mathbb{N}} Q_k$. It is also clear that $\bigcup_{k\in\mathbb{N}} Q_k \subset \bigcup_{k\in\mathbb{N}} Q_k^*$, so we have $\bigcup_{k\in\mathbb{N}} Q_k^* = \bigcup_{k\in\mathbb{N}} Q_k$. □

We can extend this lemma to obtain an additional estimate on $\{Q_k^*\}_{k\in\mathbb{N}}$.

Corollary 3.18. *Let $\{Q_k\}_{k\in\mathbb{N}}$, $\{Q_k^*\}_{k\in\mathbb{N}}$ be given as in Lemma 3.17 with scale factor $\beta \in \left(1, \frac{4}{3}\right)$. Then*

$$\left(\frac{3}{2\beta} - 1\right) \operatorname{diam} Q_k^* \leqslant d(Q_k^*, F) \leqslant \left(\frac{9}{2\beta} - 1\right) \operatorname{diam} Q_k^* \tag{3.4}$$

Proof. Consider an arbitrary $Q \in \{Q_k\}_{k\in\mathbb{N}}$ and its rescaled version Q^*. Using 3.3, we construct the largest possible Ω^- and the smallest possible Ω^+, so that we can guarantee $\Omega^- \subset \Omega \subset \Omega^+$.

We then have

$$\operatorname{dist}(x, \mathbb{R}^2 \backslash \Omega^-) = \operatorname{diam} Q,$$
$$\operatorname{dist}(x, \mathbb{R}^2 \backslash \Omega^+) = 4 \operatorname{diam} Q,$$

for all $x \in \partial Q$. Let $\delta := \operatorname{diam} Q$ so $\delta^* := \operatorname{diam} Q^* = \beta\delta$.

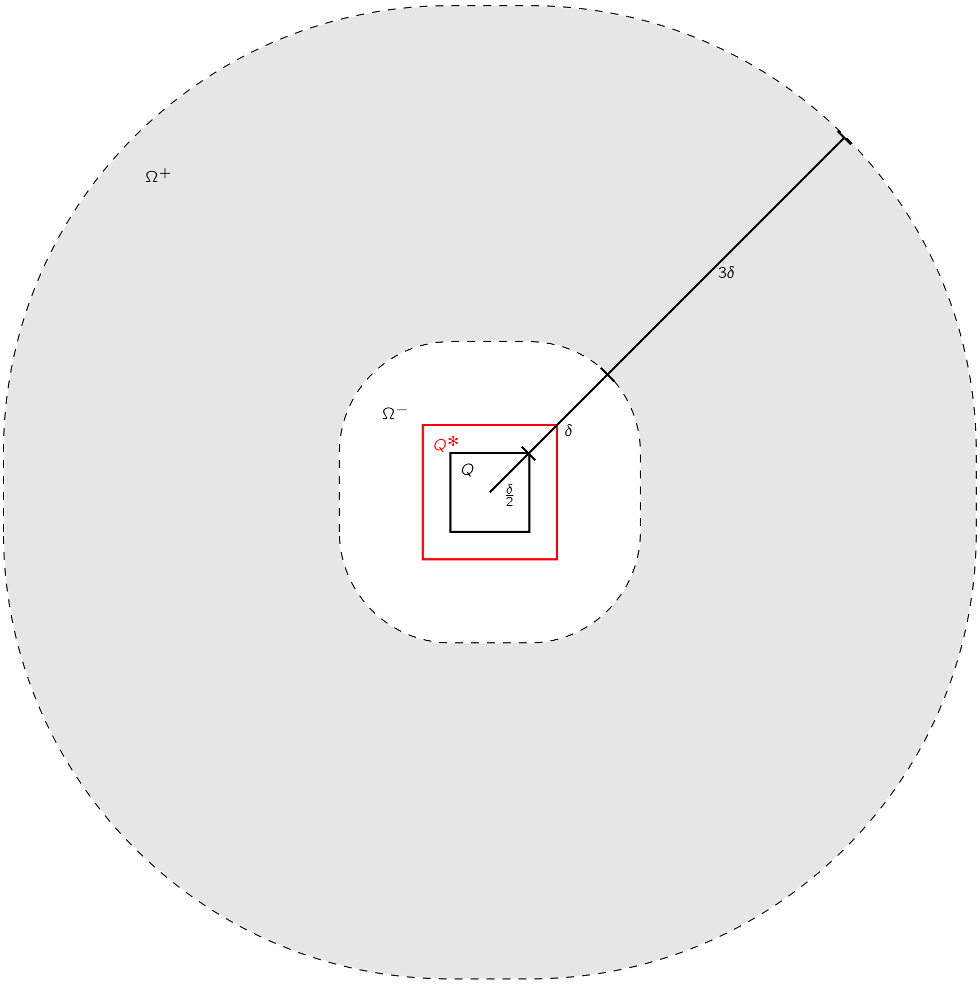

Figure 3.6: Diagram showing the square Q, its rescaled version Q^*, and the regions Ω^- and Ω^+, which are constructed to bound Ω.

Now we calculate

$$d(Q^*, F) \geqslant d(Q^*, \mathbb{R}^2 \backslash \Omega^-) = \delta - (\beta - 1)\frac{\delta}{2} = \left(\frac{3}{2} - \beta\right)\delta = \left(\frac{3}{2\beta} - 1\right)\delta^*,$$
$$d(Q^*, F) \leqslant d(Q^*, \mathbb{R}^2 \backslash \Omega^+) = 4\delta - (\beta - 1)\frac{\delta}{2} = \left(\frac{9}{2} - \beta\right)\delta = \left(\frac{9}{2\beta} - 1\right)\delta^*.$$

□

For future, reference, we shall label the constants that appear in this bound as follows:

$$\kappa_1 := \frac{3}{2\beta} - 1,$$
$$\kappa_2 := \frac{9}{2\beta} - 1.$$

Now we can use this covering lemma to prove a Calderon-Zygmund decomposition. Again, we will follow the proofs given by Lamm [29] and Stein [38], taking additional care to ensure that the constants can be bounded explicitly. The proof of the Calderon-Zygmund decomposition makes repeated use of a technique [29, p. 47 and Lemma 4.2.8] to bound certain inner product terms using the maximal function.

Lemma 3.19. *Let $f \in \mathcal{H}^1(\mathbb{R}^2)$. If $\phi \in C_c^\infty(\mathbb{R}^2)$ has support in $B_t(x)$, then*

$$|(f, \phi)| \leqslant (2t)^3 \|\nabla\phi\|_\infty f^*(z) \qquad \forall z \in B_t(x).$$

Proof. Let $z \in B_t(x)$ and define a $\psi \in \mathcal{T}$ by

$$\psi(y) = \frac{1}{2t\|\nabla\phi\|_\infty}\phi\left(z - 2ty\right).$$

Hence, we find that

$$(f * \psi_{2t})(z) = \frac{1}{(2t)^2}\int_{\mathbb{R}^2} f(y)\psi\left(\frac{z-y}{2t}\right)\mathrm{d}y = \frac{1}{(2t)^3\|\nabla\phi\|_\infty}\int_{\mathbb{R}^2} f(y)\phi(y)\,\mathrm{d}y,$$

and so

$$|(f, \phi)| \leqslant (2t)^3\|\nabla\phi\|_\infty |(f * \psi_{2t})(x)| \leqslant (2t)^3\|\nabla\phi\|_\infty f^*(z).$$

□

We can now prove the Calderon-Zygmund decomposition, calculating explicit bounds for all the required constants.

Theorem 3.20. *Let $f \in \mathcal{H}^1(\mathbb{R}^2)$ and $\lambda > 0$. Then there exists a decomposition $f = g + b$ with $b = \sum_{k=1}^\infty b_k$, and a countable family of cubes $\{C_k\}_{k\in\mathbb{N}}$ such that:*

1. *For almost every $x \in \mathbb{R}^2$ we have*
$$|g(x)| \leqslant c_1\lambda.$$
2. *Every function b_k has support in C_k, zero mean and*
$$\int_{C_k} b_k^*(x)\,\mathrm{d}x \leqslant c_2\int_{C_k} f^*(x)\,\mathrm{d}x.$$
3. *The family of cubes $\{C_k\}_{k\in\mathbb{N}}$ has the bounded intersection property and, if we set $\Omega = \bigcup_{k\in\mathbb{N}} C_k$, we have*
$$\Omega = \{x \in \mathbb{R}^2 : f^*(x) > \lambda\}.$$

Proof. Let $f \in \mathcal{H}^1(\mathbb{R}^2)$ and $\lambda > 0$. We shall prove each part separately.

Proof of (3)

We first set

$$\Omega = \{x \in \mathbb{R}^2 : f^*(x) > \lambda\},$$

and note that Ω is open [29]. We then apply Lemma 3.17 to Ω to obtain a partition of Ω into cubes $\{C'_k\}_{k\in\mathbb{N}}$. In particular, Lemma 3.17 allows us to pick constants $\alpha, \beta \in \mathbb{R}$ with $1 < \alpha < \beta < \frac{4}{3}$ such that, if $\tilde{C}_k$, C_k denote the cubes having the same centres as C'_k but scaled by factors of α, β (respectively), then $\Omega = \bigcup_{k\in\mathbb{N}} C_k$ and the family $\{C_k\}_{k\in\mathbb{N}}$ has the bounded intersection property.

Proof of (1)

Next, we consider a positive flat-top bump function $\xi \in C_c^\infty\left(\left[-\frac{a}{2}, +\frac{a}{2}\right]^2\right)$ with $\xi = 1$ on $\left[-\frac{1}{2}, +\frac{1}{2}\right]^2$, so $\|\nabla\xi\|_\infty \leqslant \frac{2}{\alpha-1}$. We then define $\xi_k \in C_c^\infty(\tilde{C}_k)$ on each cube $\tilde{C}_k$ by taking

$$\xi_k(x) = \xi\left(\frac{x - c_k}{s_k}\right), \qquad x \in \tilde{C}_k,$$

where c_k is the centre of C'_k and s_k is the side length of C'_k. Then we have $\xi_k = 1$ on C'_k and $\|\nabla\xi_k\|_\infty \leqslant \frac{2}{\alpha-1} \cdot \frac{1}{s_k}$.

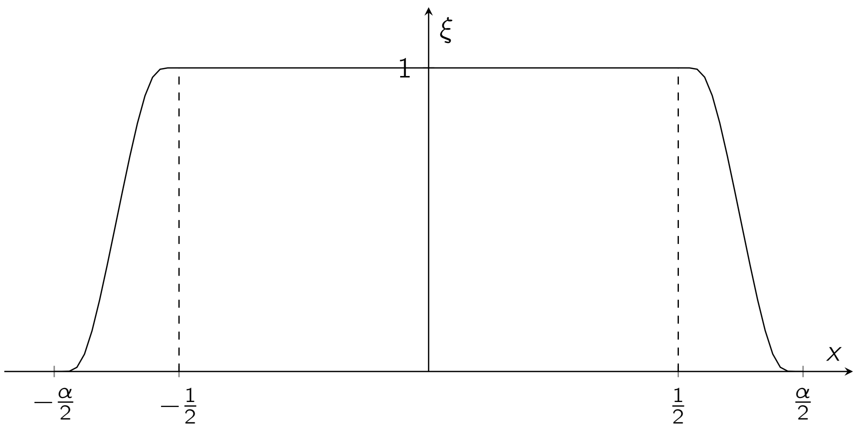

Figure 3.7: Side-on view of an example of a flat-top positive bump function ξ.

We then define a partition of unity $\{\eta_k\}_{k\in\mathbb{N}}$ on $\tilde{C}_k$ by taking

$$\eta_k(x) = \frac{\xi_k}{\sum_{j=1}^\infty \xi_j} \in C_c^\infty(\tilde{C}_k),$$

where the sum in the denominator is always finite due to the bounded intersection property of the cubes $\{\tilde{C}_k\}_{k\in\mathbb{N}}$. Furthermore, we have $\sum_{j=1}^\infty \xi_j \geqslant 1$ since the C'_k partition Ω. We then calculate

$$s_k^2 = |C'_k| = \int_{C'_k} \eta_k \,\mathrm{d}x \leqslant \int_{C_k} \eta_k \,\mathrm{d}x \leqslant |C_k| = \beta^2 s_k^2, \tag{3.5}$$

and

$$\begin{aligned}
\|\nabla\eta_k\|_\infty &= \left\| \frac{\nabla\xi_k}{\sum_{j=1}^\infty \xi_j} - \frac{\xi_k \sum_{j=1}^\infty \nabla\xi_j}{\left(\sum_{j=1}^\infty \xi_j\right)^2} \right\|_\infty \\
&\leqslant \|\nabla\xi_k\|_\infty + \|\xi_k\|_\infty \sum_{j=1}^\infty \|\nabla\xi_j\|_\infty \\
&\leqslant \frac{2}{\alpha-1}\cdot\frac{1}{s_k} + (1+N_L)\frac{2}{\alpha-1}\cdot\frac{1}{s_k} \\
&\leqslant \frac{2(2+N_L)}{(\alpha-1)s_k} =: \frac{c}{s_k}.
\end{aligned} \tag{3.6}$$

The upper bound for $\sum_{j=1}^\infty \|\nabla\xi_j\|_\infty$ is obtained by considering the worst case scenario as given in the proof of Lemma 3.17, where we pick L such that $\alpha \leqslant \beta_L$. Then $1+N_L$ of the squares overlap, and so no more than $1+N_L$ of the $\nabla\xi_j$ are non-zero.

Next, we define b_k by

$$b_k(x) := (f(x)-a_k)\eta_k(x), \qquad a_k := \frac{\int_{C_k}\eta_k(x)f(x)\,\mathrm{d}x}{\int_{C_k}\eta_k(x)\,\mathrm{d}x}.$$

Thus, b_k has support in C_k and zero mean on C_k. Then we define g by

$$g(x) := \begin{cases} f(x), & x \notin \Omega, \\ \sum_{k=1}^\infty a_k\eta_k(x), & x \in \Omega. \end{cases}$$

We can write

$$a_k = \int_{C_k} f\hat{\eta}_k\,\mathrm{d}x = (f,\hat{\eta}_k), \qquad \hat{\eta}_k := \frac{\eta_k}{\int_{C_k}\eta_k\,\mathrm{d}x},$$

and so

$$\|\nabla\hat{\eta}_k\|_\infty \leqslant \frac{\|\nabla\eta_k\|_\infty}{\int_{C_k}\eta_k\,\mathrm{d}x} \leqslant \frac{c}{s_k}\cdot\frac{1}{s_k^2} = \frac{c}{s_k^3},$$

using (3.5). Letting $\rho = (2\kappa_2+1)\frac{\beta}{\sqrt{2}}$, we observe that $C_k \subset B_{\rho s_k}(c_k)$ and $B_{\rho s_k}(c_k)\cap\Omega^c \neq \varnothing$ by (3.4). Then we use Lemma 3.19 to estimate

$$|a_k| = |(f,\hat{\eta}_k)| \leqslant (2\rho s_k)^3 \|\nabla\hat{\eta}_k\|_\infty f^*(z) \leqslant (2\rho)^3 c f^*(z) =: c_1 f^*(z), \qquad \forall z \in B_{\rho s_k}(c_k).$$

Thus, we have

$$|a_k| \leqslant c_1 f^*(x), \qquad \forall x \in C_k \subset B_{\rho s_k}(c_k), \tag{3.7}$$

$$|a_k| \leqslant c_1 f^*(z) \leqslant c_1\lambda, \qquad \forall z \in B_{\rho s_k}(c_k)\cap\Omega^c \neq \varnothing. \tag{3.8}$$

We then obtain the bound for g as follows.

$$|g(x)| = |f(x)| \leqslant f^*(x) \leqslant \lambda \leqslant c_1\lambda, \qquad \forall x \in \Omega,$$
$$|g(x)| \leqslant \sum_{k=1}^{\infty} |a_k|\, \eta_k(x) \leqslant c_1\lambda \sum_{k=1}^{\infty} \eta_k(x) = c_1\lambda, \qquad \forall x \notin \Omega.$$

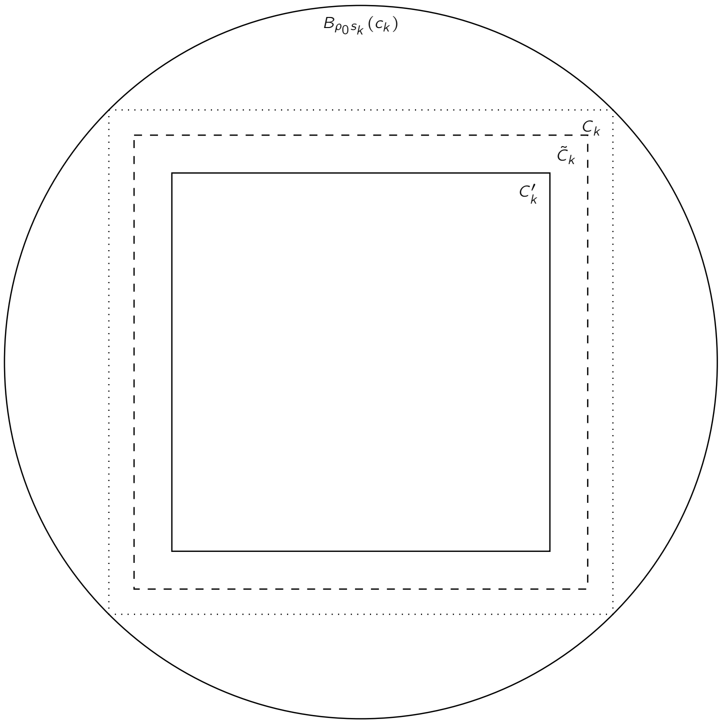

Figure 3.8: Diagram depicting the squares C_k', $\tilde{C}_k$ and C_k along with the ball $B_{\rho_0 s_k}(c_k)$.

Proof of (2)

We now fix $k \in \mathbb{N}$ and wish to bound b_k^* uniformly in k. We shall consider the cases of $x \in C_k$ and $x \notin C_k$ separately.

First, given an $x \in C_k$ and $\phi \in \mathcal{T}$, we define $\varphi_{t,\phi}$ by

$$\varphi_{t,\phi}(y) = \eta_k(y)\phi_t(x-y), \qquad t > 0.$$

Then we estimate

$$\begin{aligned}\left\|\nabla\varphi_{t,\phi}\right\|_\infty &\leqslant \|\nabla\eta_k\|_\infty \cdot \|\phi_t\|_\infty + \|\eta_k\|_\infty \|\nabla\phi_t\|_\infty \\ &\leqslant \frac{\|\nabla\eta_k\|_\infty}{t^2} + \|\nabla\phi_t\|_\infty \\ &\leqslant \frac{c}{s_k t^2} + \frac{1}{t^3}.\end{aligned}$$

Now we consider two possible cases:

- If $2t \leqslant s_k$, then we have $\left\|\nabla\varphi_{t,\phi}\right\|_\infty \leqslant \left(\frac{c}{2}+1\right) t^{-3}$. Since $\varphi_{t,\phi}$ is smooth with support in $B_t(x)$, we can again use Lemma 3.19 to obtain

$$\sup_{\phi\in\mathcal{T}}\sup_{0<2t<s_k}\left|\int_{\mathbb{R}^2}\varphi_{t,\phi}(y)f(y)\,\mathrm{d}y\right| \leqslant \sup_{0<2t<s_k}(2t)^3\cdot\left(\frac{c}{2}+1\right)t^{-3}f^*(z) = 4(c+2)f^*(z),$$

for any $z \in B_t(x)$. In particular, this inequality holds for $z = x \in C_k$.

- If $2t > s_k$, then we have $\left\|\nabla\varphi_{t,\phi}\right\|_\infty \leqslant 4\,(c+2)\,s_k^{-3}$. Since $\varphi_{t,\phi}$ is smooth with support in $B_{\rho_0 s_k}(x)$, where $\rho_0 = \frac{\beta}{\sqrt{2}}$, we can again use Lemma 3.19 to obtain

$$\sup_{\phi\in\mathcal{T}}\sup_{2t>s_k}\left|\int_{\mathbb{R}^2}\varphi_{t,\phi}(y)f(y)\,\mathrm{d}y\right| \leqslant (2\rho_0 s_k)^3\cdot 4\,(c+2)\,s_k^{-3}f^*(z) = 4(c+2)(2\rho_0)^3f^*(z),$$

for any $z \in B_{\rho_0 s_k}(x)$. In particular, this inequality holds for $z = x \in C_k$.

Hence, we have

$$(f\eta_k)^*(x) = \sup_{\phi\in\mathcal{T}}\sup_{t>0}\left|\int_{\mathbb{R}^2}\varphi_{t,\phi}(y)f(y)\,\mathrm{d}y\right| \leqslant 4(c+2)(2\rho_0)^3f^*(x), \tag{3.9}$$

since $2\rho_0 = \beta\sqrt{2} > 1$. We also note that

$$\eta_k^*(x) = \sup_{\phi\in\mathcal{T}}\sup_{t>0}\left|\int_{\mathbb{R}^2}\varphi_{t,\phi}(y)\,\mathrm{d}y\right| \leqslant \sup_{\phi\in\mathcal{T}}\sup_{t>0}\int_{B_t(x)}\frac{1}{t^2}\,\mathrm{d}y = \pi, \tag{3.10}$$

and so

$$b_k^*(x) \leqslant (f\eta_k)^*(x) + |a_k|\,\eta_k^*(x) \leqslant \left(4(c+2)(2\rho_0)^3 + c_1\pi\right)f^*(x), \tag{3.11}$$

using (3.7).

Now suppose $x \notin C_k$ and again let $\phi \in \mathcal{T}$. This time we define $\psi_{t,\phi}$ by

$$\psi_{t,\phi}(y) = \eta_k(y)(\phi_t(x-y) - \phi_t(x-c_k)), \qquad t > 0.$$

Take $t \geqslant |x-c_k|$ so the supports of η_k and $y \mapsto \phi_t(x-y)$ intersect (see Figure 3.2.2.). Then, using the mean value theorem, we estimate

$$\left|\psi_{t,\phi}(y)\right| \leqslant |\eta_k(y)|\cdot\|\nabla\phi_t\|_\infty\cdot|y-c_k| \leqslant \frac{|y-c_k|}{t^3} \leqslant \frac{\beta s_k}{|x-c_k|^3\sqrt{2}}, \tag{3.12}$$

for $y \in C_k$ and $\psi_{t,\phi}(y) = 0$ for $y \notin C_k$. Again using the mean value theorem, we also have

$$\begin{aligned}\left|\nabla\psi_{t,\phi}(y)\right| &\leqslant |\eta_k(y)|\cdot|\nabla\phi_t(x-y)| + |\nabla\eta_k(y)|\cdot|\phi_t(x-y)-\phi_t(x-c_k)| \\ &\leqslant \frac{1}{t^3} + \frac{c}{s_k}\cdot\|\nabla\phi_t\|_\infty\cdot|y-c_k| \\ &\leqslant \frac{\sqrt{2}+\beta c}{t^3\sqrt{2}},\end{aligned}$$

for $y \in C_k$. Furthermore, since $\psi_{t,\phi}$ is smooth with support in $B_{\rho s_k}(c_k)$, we can again use Lemma 3.19 with $z \in B_{\rho s_k}(c_k) \cap \Omega^c \neq \varnothing$ to obtain

$$\left|(\psi_{t,\phi}, f)\right| \leqslant (2\rho s_k)^3 \cdot \left(\frac{\sqrt{2}+\beta c}{t^3\sqrt{2}}\right) f^*(z) \leqslant \frac{(2\rho s_k)^3(\sqrt{2}+\beta c)\lambda}{\sqrt{2}\,|x-c_k|^3}. \tag{3.13}$$

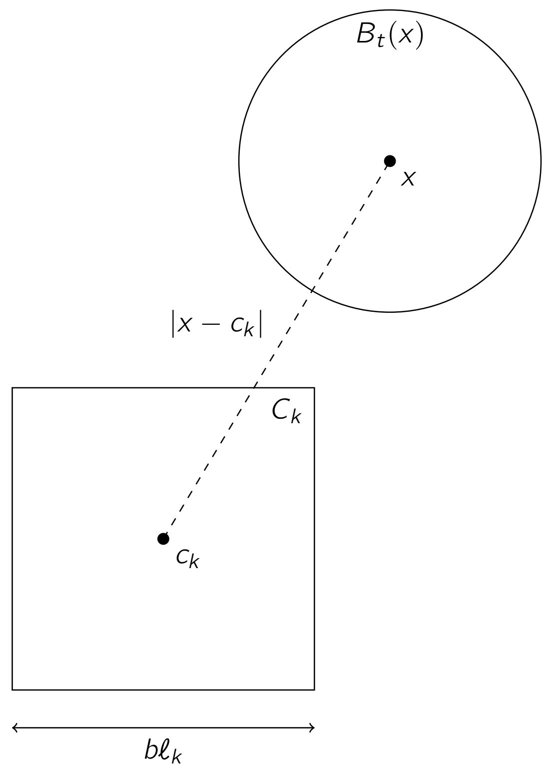

Figure 3.9: Diagram showing the square C_k and the ball $B_t(x)$, with $x \notin C_k$. Observe that the two sets will intersect when $t \geqslant |x - c_k|$.

In order to bound b_k^*, we will estimate the following integrals:

$$I_1(x) = \int_{\mathbb{R}^2} (\phi_t(x-y) - \phi_t(x-c_k)) f(y)\eta_k(y)\,\mathrm{d}y,$$
$$I_2(x) = \int_{\mathbb{R}^2} (\phi_t(x-y) - \phi_t(x-c_k)) a_k\eta_k(y)\,\mathrm{d}y.$$

For I_1, we have

$$|I_1(x)| = \left|(\psi_{t,\phi}, f)\right| \leqslant \frac{(2\rho s_k)^3(\sqrt{2}+\beta c)\lambda}{\sqrt{2}\,|x-c_k|^3},$$

using (3.13). For I_2, we have

$$|I_2(x)| \leqslant |a_k| \cdot |C_k| \cdot \left\|\psi_{t,\phi}\right\|_\infty \leqslant \frac{c_1(\beta s_k)^3\lambda}{|x-c_k|^3\sqrt{2}},$$

using (3.8) and (3.12). Hence,

$$b_k^*(x) \leqslant \sup_{\phi\in\mathcal{T}} \sup_{t\geqslant|x-c_k|} |I_1 - I_2| \leqslant \frac{1}{\sqrt{2}}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\frac{\lambda s_k^3}{|x-c_k|^3}. \tag{3.14}$$

Note that we do not have to consider the case of $t < |x - c_k|$ as here $\psi_{t,\phi} \propto \eta_k$ and so $I_1 - I_2 \propto \int_{\mathbb{R}^2} b_k \, \mathrm{d}x = 0$.

Overall, we have

$$b_k^*(x) \leqslant \begin{cases} \left(4(c+2)(\beta\sqrt{2})^3 + c_1\pi\right) f^*(x), & x \in C_k, \\ \frac{1}{\sqrt{2}}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\frac{\lambda s_k^3}{|x-c_k|^3}, & x \notin C_k, \end{cases}$$

by (3.11) and (3.14). Now we estimate

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus C_k} b_k^*(x)\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^2\setminus C_k} \frac{1}{\sqrt{2}}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\frac{\lambda s_k^3}{|x-c_k|^3}\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^2\setminus B_{\beta s_k/\sqrt{2}}(c_k)} \frac{1}{\sqrt{2}}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\frac{\lambda s_k^3}{|x-c_k|^3}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2}}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\int_0^{2\pi}\int_{\beta s_k/\sqrt{2}}^{\infty} \frac{\lambda s_k^3}{r^3} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{\sqrt{2}}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\left(2\pi\lambda s_k^3\frac{\sqrt{2}}{\beta s_k}\right) \\
&= \frac{2\pi}{\beta^3}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\lambda(\beta s_k)^2 \\
&= \frac{2\pi}{\beta^3}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\lambda\,|C_k| \\
&\leqslant \frac{2\pi}{\beta^3}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\int_{C_k} f^*(x)\,\mathrm{d}x,
\end{aligned}$$

and then

$$\begin{aligned}
&\int_{\mathbb{R}^2} b_k^*(x)\,\mathrm{d}x = \int_{C_k} b_k^*(x)\,\mathrm{d}x + \int_{\mathbb{R}^2\setminus C_k} b_k^*(x)\,\mathrm{d}x \\
&\leqslant \left(4(c+2)(\beta\sqrt{2})^3 + c_1\pi + \frac{2\pi}{\beta^3}\left((2\rho)^3(\sqrt{2}+\beta c) + c_1\beta^3\right)\right)\int_{C_k} f^*(x)\,\mathrm{d}x =: c_2\int_{C_k} f^*(x)\,\mathrm{d}x.
\end{aligned}$$

□

3.2.3. Duality via Atomic Decomposition

Next, we will show that functions in $\mathcal{H}^1$ can be decomposed using $\mathcal{H}^1$-atoms and that we can relate this decomposition to the $\mathcal{H}^1$ norm. This is a fundamental result in Harmonic analysis with many proofs given in the literature [29, 38]. We will improve upon these results by giving a bound for the constant that we will refer to as C_*.

Theorem 3.21. *Let $f \in \mathcal{H}^1(\mathbb{R}^2)$. Then there exists a countable family $\{a_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}^1$-atoms and coefficients $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, such that*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k,$$

where the convergence of the sum is with respect to the $\mathcal{H}^1$ norm. Furthermore, we have

$$\sum_{k=1}^{\infty} |\lambda_k| \leqslant C_* \|f\|_{\mathcal{H}^1}.$$

Proof. For each $j \in \mathbb{Z}$, we apply Theorem 3.20 to $f \in \mathcal{H}^1(\mathbb{R}^2)$ and $\lambda = 2^j > 0$. We then get a countable family of cubes $\{C_k^j\}_{k\in\mathbb{N}}$ in a set Ω^j and a decomposition

$$f(x) = g^j(x) + b^j(x) = g^j(x) + \sum_{k=1}^{\infty} b_k^j(x),$$

for each $j \in \mathbb{Z}$. By the definition of the Ω^j, we observe that $\Omega^{j+1} \subset \Omega^j$ for every $j \in \mathbb{Z}$. Now we estimate

$$\begin{aligned}
\left\|f - g^j\right\|_{\mathcal{H}^1} = \left\|b^j\right\|_{\mathcal{H}^1} &\leqslant \sum_{k=1}^{\infty} \left\|b_k^j\right\|_{\mathcal{H}^1} \\
&\leqslant c_2 \sum_{k=1}^{\infty} \int_{C_k^j} f^*(x)\,\mathrm{d}x \\
&\leqslant c_2(1 + N_L) \int_{\Omega^j} f^*(x)\,\mathrm{d}x \\
&= c_2(1 + N_L) \int_{\{x\in\mathbb{R}^2 : f^*(x) > 2^j\}} f^*(x)\,\mathrm{d}x,
\end{aligned}$$

as no more than $1 + N_L$ squares overlap by the bounded intersection property of $\Omega^j = \bigcup_{k\in\mathbb{N}} C_k^j$. Since $f \in \mathcal{H}^1(\mathbb{R}^2)$, we know that $f^* \in L^1(\mathbb{R}^2)$, and so

$$\lim_{j\to+\infty} g^j = f,$$

in $\mathcal{H}^1$. By the first property of Theorem 3.20, we have that $\left|g^j(x)\right| \leqslant c_1 \cdot 2^j$, and so

$$\lim_{j\to-\infty} g^j = 0,$$

in the sense of distributions. Thus, we can write

$$\begin{aligned}
f &= \lim_{N\to+\infty} g^N - \lim_{N\to-\infty} g^N \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} (g^{j+1} - g^j) + \lim_{N\to\infty} \sum_{j=-N}^{0} (g^{j+1} - g^j) \\
&= \sum_{j\in\mathbb{Z}} (g^{j+1} - g^j),
\end{aligned}$$

where the convergence is, again, in the sense of distributions. We also have

$$g^{j+1} - g^j = (f - b^{j+1}) - (f - b^j) = b^j - b^{j+1},$$

by (3.2.3.) so $g^{j+1} - g^j$ is supported in $\Omega^j \supset \Omega^{j+1}$ and

$$\left|g^{j+1}(x) - g^j(x)\right| \leqslant \left|g^{j+1}(x)\right| + \left|g^j(x)\right| \leqslant c_1 \cdot 2^{j+1} + c_1 \cdot 2^j = 3c_1 \cdot 2^j, \qquad \forall x \in \mathbb{R}^2,$$

using the first property of Theorem 3.20. Now, for each $j \in \mathbb{Z}$, we recall the partition of unity $\eta_k^j \in C_c^\infty(C_k^j)$ as given in the proof of Theorem 3.20. We can then write

$$f = \sum_{j\in\mathbb{Z}} \sum_{k=1}^{\infty} (g^{j+1} - g^j)\eta_k^j.$$

Letting $B_k^j = B_{\rho_0 s_k^j}(c_k^j)$, we define real numbers

$$\lambda_{j,k} := 3c_1 \cdot 2^j \left|B_k^j\right|.$$

Also letting $A_{j,k} = (g^{j+1} - g^j)\eta_k^j$, we define functions

$$a_{j,k} := \frac{A_{j,k}}{\lambda_{j,k}},$$

so the support of $A_{j,k}$ (hence, $a_{j,k}$) lies in $B_k^j \supset C_k^j$ and

$$\left\|a_{j,k}\right\|_\infty \leqslant \frac{\left\|\eta_k^j\right\|_\infty}{\lambda_{j,k}} \left\|g^{j+1} - g^j\right\|_\infty \leqslant \frac{1}{\left|B_k^j\right|}.$$

We then have

$$f = \sum_{j\in\mathbb{Z}} \sum_{k=1}^{\infty} \lambda_{j,k} a_{j,k},$$

and

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \sum_{k=1}^{\infty} \left|\lambda_{j,k}\right| &\leqslant 3c_1 \sum_{j\in\mathbb{Z}} 2^j \sum_{k=1}^{\infty} \left|B_k^j\right| \\
&= 3c_1 \sum_{j\in\mathbb{Z}} 2^j \sum_{k=1}^{\infty} \pi\rho_0^2 (s_k^j)^2 \\
&= 3\pi c_1 \rho_0^2 \sum_{j\in\mathbb{Z}} 2^j \left|\Omega^j\right| \\
&= 3\pi c_1 \rho_0^2 \sum_{j\in\mathbb{Z}} 2^j \left|\{x \in \mathbb{R}^2 : f^*(x) > 2^j\}\right| \\
&= 3\pi c_1 \rho_0^2 \sum_{j\in\mathbb{Z}} \left(\sum_{k=-\infty}^{j} 2^k\right) \left|\{x \in \mathbb{R}^2 : 2^{j+1} \geqslant f^*(x) > 2^j\}\right|.
\end{aligned}$$

The final line is due to the fact that every set of the form $\{x \in \mathbb{R}^2 : 2^{j+1} \geqslant f^*(x) > 2^j\}$ gets counted multiple times in the sum, weighted by 2^k for each integer $k \leqslant j$. Continuing with the calculation, we find that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{k=1}^{\infty}\left|\lambda_{j,k}\right| &\leqslant 3\pi c_1\rho_0^2\sum_{j\in\mathbb{Z}}2^{j+1}\left|\{x\in\mathbb{R}^2 : 2^{j+1}\geqslant f^*(x) > 2^j\right| \\
&= 6\pi c_1\rho_0^2\sum_{j\in\mathbb{Z}}2^{j}\left|\{x\in\mathbb{R}^2 : 2^{j+1}\geqslant f^*(x) > 2^j\right| \\
&\leqslant 6\pi c_1\rho_0^2\int_{\mathbb{R}^2} f^*(x)\,\mathrm{d}x = 6\pi c_1\rho_0^2\left\|f\right\|_{\mathcal{H}^1}.
\end{aligned}$$

If we additionally had $\int_{\mathbb{R}^2} a_{j,k}(x)\,\mathrm{d}x = 0$, then we would be done. However, this does not necessarily hold, and so we will need to modify the decomposition. First, we define

$$\tilde{A}_{j,k}(x) := b_k^j(x) - \sum_{\ell=1}^{\infty} b_\ell^{j+1}(x)\eta_k^j(x) + \sum_{\ell=1}^{\infty} c_{\ell,k}\eta_\ell^{j+1}(x), \qquad c_{\ell,k} := \frac{\int_{\mathbb{R}^2} b_\ell^{j+1}(x)\eta_k^j(x)\,\mathrm{d}x}{\int_{\mathbb{R}^2}\eta_\ell^{j+1}(x)\,\mathrm{d}x}.$$

Then, using the second property of Theorem 3.20, we calculate

$$\int_{\mathbb{R}^2}\tilde{A}_{j,k}\,\mathrm{d}x = \int_{\mathbb{R}^2} b_k^j\,\mathrm{d}x - \sum_{\ell=1}^{\infty}\int_{\mathbb{R}^2} b_\ell^{j+1}\eta_k^j\,\mathrm{d}x + \sum_{\ell=1}^{\infty}\int_{\mathbb{R}^2} b_\ell^{j+1}\eta_k^j\,\mathrm{d}x = 0,$$

and

$$\sum_{k=1}^{\infty} c_{\ell,k} = \frac{\int_{\mathbb{R}^2} b_\ell^{j+1}\,\mathrm{d}x}{\int_{\mathbb{R}^2}\eta_\ell^{j+1}\,\mathrm{d}x} = 0,$$

so

$$\sum_{k=1}^{\infty}\tilde{A}_{j,k} = b^j - b^{j+1} = \sum_{k=1}^{\infty} A_{j,k}.$$

Thus, we have

$$f = \sum_{j\in\mathbb{Z}}\sum_{k=1}^{\infty}\tilde{A}_{j,k}.$$

Next we observe that, for each $\ell \in \mathbb{N}$, $b_{\ell,k} \neq 0$ requires that $C_\ell^{j+1} \cap C_k^j$ is non-empty, and so

$$\begin{aligned}
\operatorname{supp}\tilde{A}_{j,k} &\subset \operatorname{supp} b_k^j \cup \left(\operatorname{supp} b^{j+1} \cap \operatorname{supp}\eta_k^j\right) \cup \left(\bigcup_{\ell\in\mathbb{N}:C_\ell^j\cap C_k^j\neq\varnothing}\operatorname{supp}\eta_\ell^{j+1}\right), \\
&\subset C_k^j \cup \left(\Omega^{j+1}\cap C_k^j\right) \cup \left(\bigcup_{\ell\in\mathbb{N}:C_\ell^j\cap C_k^j\neq\varnothing} C_\ell^{j+1}\right), \\
&\subset \tilde{B}_k^j := B_{\tilde{\rho}s_k^j}(c_k^j), \qquad \tilde{\rho} = \left(1 + 2\frac{\kappa_2}{\kappa_1}\right)\frac{\beta}{\sqrt{2}}
\end{aligned}$$

The final inclusion comes from studying the relative sizes of the C^j_k and C^{j+1}_ℓ using (3.4) and Lemma 3.16. We have

$$\kappa_1 \operatorname{diam} C^{j+1}_\ell \leqslant d(C^{j+1}_\ell, (\Omega^{j+1})^c) \leqslant d(C^{j+1}_\ell, (\Omega^j)^c),$$

since $\Omega^j \supset \Omega^{j+1}$. Hence,

$$\kappa_1 \operatorname{diam} C^{j+1}_\ell \leqslant \operatorname{diam} C^j_k + d(C^j_k, (\Omega^j)^c) \leqslant \kappa_2 \operatorname{diam} C^j_k,$$

so $s^{j+1}_\ell \leqslant \frac{\kappa_2}{\kappa_1} s^j_k$. To obtain boundedness, we observe that η^j_k is smooth with support in $\tilde{B}^j_k$. Thus, using Lemma 3.19, we have

$$|c_{\ell,k}| \leqslant \frac{(2\tilde{\rho} s^j_k)^3 \left\|\nabla \eta^j_k\right\|_\infty}{\left|C^{j+1}_\ell\right|} (b^{j+1}_\ell)^*(z) \leqslant \frac{(2\tilde{\rho})^3 (s^j_k)^2 c}{(\beta s^{j+1}_\ell)^2} (b^{j+1}_\ell)^*(z), \qquad \forall z \in \tilde{B}^j_k.$$

Now consider $z \in \tilde{B}^j_k \cap (C^{j+1}_\ell)^c \neq \varnothing$ such that $\left|z - c^{j+1}_\ell\right| \geqslant \frac{\beta\kappa_2}{\sqrt{2}\kappa_1} s^j_k$. We then have

$$\begin{aligned} |c_{\ell,k}| &\leqslant \frac{(2\tilde{\rho})^3 (s^j_k)^2 c}{(\beta s^{j+1}_\ell)^2} \cdot \frac{1}{\sqrt{2}} \left((2\rho)^3(\sqrt{2} + \beta c) + c_1\beta^3\right) \frac{2^{j+1}(s^{j+1}_\ell)^3}{\left|z - c^{j+1}_\ell\right|^3} \\ &\leqslant \frac{2(2\tilde{\rho})^3 c}{\beta^5} \left((2\rho)^3(\sqrt{2} + \beta c) + c_1\beta^3\right) \left(\frac{\kappa_1}{\kappa_2}\right)^2 2^j =: \tilde{c} \cdot 2^j, \end{aligned}$$

using (3.14).

Now we can estimate the $\tilde{A}_{j,k}$. We first write

$$\begin{aligned} \tilde{A}_{j,k} &= (f - a^j_k)\eta^j_k - \sum_{\ell=1}^\infty (f - a^{j+1}_\ell)\eta^{j+1}_\ell \eta^j_k + \sum_{\ell=1}^\infty c_{\ell,k}\eta^{j+1}_\ell \\ &= f\left(1 - \sum_{\ell=1}^\infty \eta^{j+1}_\ell\right)\eta^j_k - a^j_k\eta^j_k + \sum_{\ell=1}^\infty a^{j+1}_\ell \eta^{j+1}_\ell \eta^j_k + \sum_{\ell=1}^\infty c_{\ell,k}\eta^{j+1}_\ell \\ &= f\eta^j_k \chi_{(\Omega^{j+1})^c} - a^j_k\eta^j_k + g^{j+1}\eta^j_k\chi_{\Omega^{j+1}} + \sum_{\ell=1}^\infty c_{\ell,k}\eta^{j+1}_\ell, \end{aligned}$$

and so

$$\left|\tilde{A}_{j,k}\right| \leqslant 2^{j+1} + c_1 \cdot 2^j + c_1 \cdot 2^{j+1} + \tilde{c} \cdot 2^j = (2 + 3c_1 + \tilde{c}) =: C \cdot 2^j,$$

using (3.8) and the first property of Theorem 3.20. Then we define

$$\tilde{a}_{j,k} := \frac{\tilde{A}_{j,k}}{\tilde{\lambda}_{j,k}}, \qquad \tilde{\lambda}_{j,k} := C \cdot 2^j \left|\tilde{B}^j_k\right|,$$

and observe that:

- The $\tilde{a}_{j,k}$ have support in $\tilde{B}^j_k$.

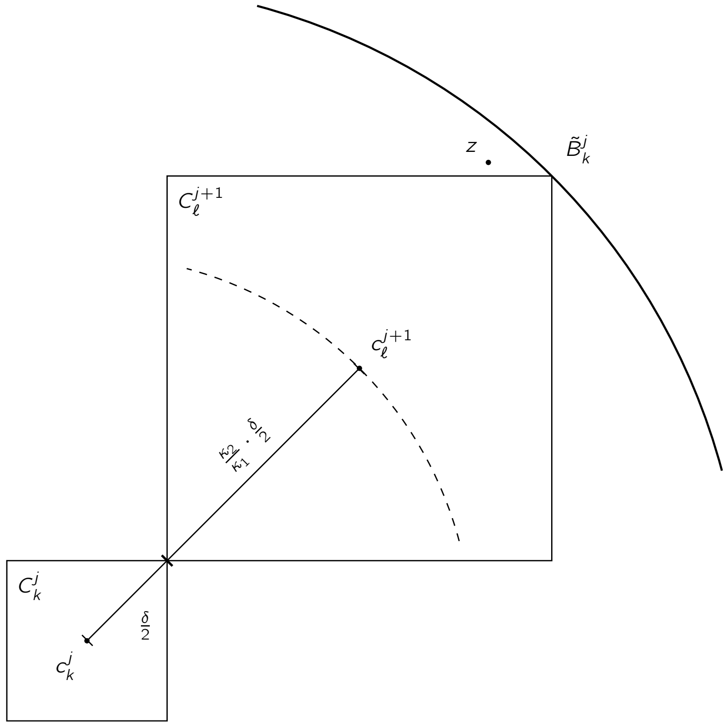

Figure 3.10: Diagram depicting an arrangement of C_k^j and C_ℓ^{j+1} with c_k^j and c_ℓ^{j+1} maximally spaced. Note that they still both fit inside the ball $\tilde{B}_k^j$ and we can pick a point $z \in \tilde{B}_k^j \cap (C_\ell^{j+1})^c$ such that $\left|z - c_\ell^{j+1}\right| \geqslant \frac{\kappa_2}{\kappa_1} \cdot \frac{\delta}{2}$where $\delta = \operatorname{diam} C_k^j$ (consider z near the corner of C_ℓ^{j+1}).

- They have zero mean.
- They are bounded with $\left|\tilde{a}_{j,k}\right| \leqslant \frac{1}{\left|\tilde{B}_{j,k}\right|}$.
- We have the decomposition

$$f = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} \tilde{\lambda}_{j,k} \tilde{a}_{j,k} .$$

Finally, we calculate

$$\sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} \left|\tilde{\lambda}_{j,k}\right| = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} \frac{C}{3c_1} \left(1 + 2\frac{\kappa_2}{\kappa_1}\right)^2 \left|\lambda_{j,k}\right| \leqslant 2\pi C \left(1 + 2\frac{\kappa_2}{\kappa_1}\right)^2 \rho_0^2 \left\|f\right\|_{\mathcal{H}^1} .$$

□

Now that we have the atomic decomposition of $\mathcal{H}^1$ functions, showing the duality between BMO and $\mathcal{H}^1$ is almost trivial.

Theorem 3.22. *The space BMO is the dual space of $\mathcal{H}^1$ and, in particular,*

$$\left|\int_{\mathbb{R}^2} f g \, \mathrm{d}x\right| \leqslant C_* \left\|f\right\|_{\mathcal{H}^1} \cdot [g]_{\mathrm{BMO}} \qquad \forall f \in \mathcal{H}^1, g \in \mathrm{BMO} .$$

Proof. We start by taking the atomic decomposition of f, writing $f = \sum_k \lambda_k a_k$. Then we calculate

$$\int_{\mathbb{R}^2} fg \, \mathrm{d}x = \sum_k \lambda_k \int_{B_k} a_k(g - g_{B_k}) \, \mathrm{d}x,$$

where B_k is the ball in which a_k has support. We then estimate

$$\left| \int_{\mathbb{R}^2} a_k(g - g_{B_k}) \, \mathrm{d}x \right| \leqslant \fint_{B_k} |g - g_{B_k}| \, \mathrm{d}x \leqslant [g]_{\mathrm{BMO}} \, .$$

Hence,

$$\left| \int_{\mathbb{R}^2} fg \, \mathrm{d}x \right| \leqslant \sum_k |\lambda_k| \cdot [g]_{\mathrm{BMO}} \leqslant C_* \, \|f\|_{\mathcal{H}^1} \cdot [g]_{\mathrm{BMO}} \, .$$

□

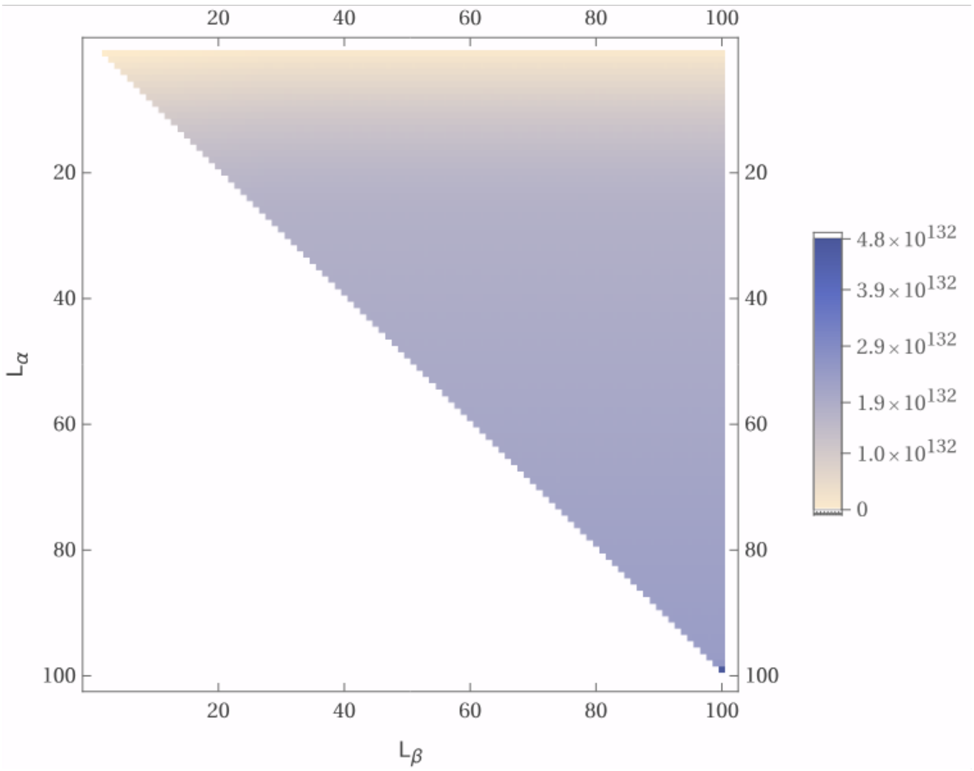

Figure 3.11: Plot showing the value of C_* for various values of L_α and L_β.

The constant C_* is entirely determined by the choice of $\alpha, \beta \in \left(1, \frac{4}{3}\right)$ in Theorem 3.20. Due to the layered construction in Lemma 3.17, we pick α and β by specifying $L_\alpha, L_\beta \in \mathbb{N}$ and taking $\alpha = \beta_{L_\alpha}$ and $\beta = \beta_{L_\beta}$. To ensure $\alpha < \beta$, we shall also take $L_\alpha < L_\beta$. Trying different values for L_α and L_β, it becomes clear that taking lower values for L_α and L_β allows us to obtain a smaller C_* (see figure 3.11). This is somewhat counter-intuitive and is likely due to a lack of optimality in some of the estimates we have used.

Instead, we consider only $L = 1$ and take $\alpha = \beta = \beta_1 = \frac{20}{17}$. We then obtain

$$C_* = \frac{1797408}{129044037529}\left(4219775450749582585 + 6067062115609586\sqrt{2}\right)\pi \approx 1.85025 \times 10^{14}$$

It should be noted that this constant could most likely be reduced by improving the estimates used in the duality proof and its dependencies. In particular, optimising the layout used in the proof of Lemma 3.17 to reduce the growth rate of N_L would lead to a much smaller value for C_*.

3.3. Examples

In this section, we apply Theorem 3.1 to some one parameter families of pressure functions. This will allow us to derive some sufficient conditions, given in terms of parameter bounds, for which the excess will be non-negative. Now that we have estimated all the required constants, we can write Theorem 3.1 as

$$[\mathsf{p}]_{\mathrm{BMO}} \leqslant C \quad \Rightarrow \quad \mathbb{E}_{\mathsf{p}} \geqslant 0$$

with $C \approx 1.09751 \times 10^{-17}$.

3.3.1. Monomial Pressure Functions

We first consider an example of a bounded pressure function $\mathsf{p} : B \to \mathbb{R}$, given by

$$\mathsf{p}(x) = |x|^{\sigma}, \qquad x \in B \subset \mathbb{R}^2,$$

for a constant $\sigma > 0$. We then seek sufficient conditions on λ for the excess $\mathbb{E}_{\lambda \mathsf{p}}$ to be non-negative. We will take advantage of the additional regularity of p to bound $[\mathsf{p}]_{\mathrm{BMO}}$. If $\sigma \geqslant 1$, then $\mathsf{p} \in W^{1,\infty}$ and we can estimate

$$\begin{aligned} \fint_{B'} |\mathsf{p}(x) - \mathsf{p}_{B'}| \,\mathrm{d}x &= \fint_{B'} |\mathsf{p}(x) - \mathsf{p}(y)| \,\mathrm{d}x \\ &\leqslant \fint_{B'} |\nabla \mathsf{p}(\xi)| \cdot |x - y| \,\mathrm{d}x \\ &\leqslant \mathrm{diam}(B') \left\| \nabla \mathsf{p}|_{B'} \right\|_{\infty} \leqslant \mathrm{diam}(B) \left\| \nabla \mathsf{p} \right\|_{\infty} = 2\sigma, \qquad \forall B' \subset B, \end{aligned}$$

using the mean-value theorem with $y, \xi \in B'$. Hence,

$$[\mathsf{p}]_{\mathrm{BMO}} \leqslant 2\sigma, \qquad \sigma \geqslant 1.$$

However, if $\sigma < 1$, then $\mathsf{p} \notin W^{1,\infty}$, but we still have $\mathsf{p} \in L^{\infty}$. We instead estimate

$$\begin{aligned} \fint_{B'} |\mathsf{p}(x) - \mathsf{p}_{B'}| \,\mathrm{d}x &\leqslant \fint_{B'} |\mathsf{p}(x)| + |\mathsf{p}_{B'}| \,\mathrm{d}x \\ &= \fint_{B'} |\mathsf{p}(x)| \,\mathrm{d}x + \left| \fint_{B'} \mathsf{p}(x) \,\mathrm{d}x \right| \\ &\leqslant \fint_{B'} |\mathsf{p}(x)| \,\mathrm{d}x + \fint_{B'} |\mathsf{p}(x)| \,\mathrm{d}x \leqslant 2 \left\| \mathsf{p}|_{B'} \right\|_{\infty} \leqslant 2 \left\| \mathsf{p} \right\|_{\infty} = 2, \end{aligned}$$

and so

$$[\mathsf{p}]_{\mathrm{BMO}} \leqslant 2, \qquad \sigma > 0.$$

In this case, we can see that we get no improvement using the $W^{1,\infty}$ estimate. A sufficient condition for $\mathbb{E}_{\lambda\mathsf{p}} \geqslant 0$ is

$$\lambda \leqslant \lambda_* := \frac{C}{[\mathsf{p}]_{\mathrm{BMO}}} \approx 5.48753 \times 10^{-18}.$$

3.3.2. Logarithmic Pressure Functions

Now let $\mathsf{p} : B \to \mathbb{R}$ be given by

$$\mathsf{p}(x) = \log(|x|) \qquad x \in B \subset \mathbb{R}^2.$$

We first calculate $\mathsf{p}_{B_r(x)}$ for some ball $B_r(x) \subset B$. However, due to radial symmetry, we need only consider $x = te_1$ for $t \in [0,1]$. We also observe that

$$B_r(te_1) \subset B \quad \iff \quad t + r \leqslant 1.$$

Then

$$\begin{aligned} \mathsf{p}_{B_r(te_1)} &= \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \log\left(|te_1 + \tilde{r} e_r(\theta|\right) \tilde{r} \,\mathrm{d}\theta \,\mathrm{d}\tilde{r} \\ &= \frac{1}{2\pi r^2} \int_0^r \int_0^{2\pi} \log\left(t^2 + 2t\tilde{r}\cos(\theta) + \tilde{r}^2\right) \tilde{r} \,\mathrm{d}\theta \,\mathrm{d}\tilde{r}. \end{aligned}$$

We then take out a factor of t^2 and change variables using $s = \frac{\tilde{r}}{t}$ to obtain

$$\mathsf{p}_{B_r(te_1)} = \frac{t^2}{2\pi r^2} \int_0^{r/t} \int_0^{2\pi} \log\left(1 + 2s\cos(\theta) + s^2\right) s \,\mathrm{d}\theta \,\mathrm{d}s + \frac{1}{2}\log\left(t^2\right).$$

We now consider the inner integral

$$\int_0^{2\pi} \log\left(1 + 2s\cos(\theta) + s^2\right) \,\mathrm{d}\theta = I(s) + I(-s),$$

where

$$I(s) := \int_0^{\pi} \log\left(1 + 2s\cos(\theta) + s^2\right) \,\mathrm{d}\theta, \qquad s \in \mathbb{R}.$$

By substituting $\theta \mapsto \pi - \theta$, we observe that $I(s) = I(-s)$. We also have

$$\begin{aligned} I(s) + I(-s) &= \int_0^{\pi} \log\left(\left(1 + 2s\cos(\theta) + s^2\right)\left(1 - 2s\cos(\theta) + s^2\right)\right) \,\mathrm{d}\theta \\ &= \int_0^{\pi} \log\left((1 + s^2)^2 - (2s\cos(\theta))^2\right) \,\mathrm{d}\theta \\ &= \int_0^{\pi} \log\left(1 - 2s^2\cos(2\theta) + s^4\right) \,\mathrm{d}\theta \\ &= \frac{1}{2} \int_0^{2\pi} \log\left(1 - 2s^2\cos(\theta) + s^4\right) \,\mathrm{d}\theta = \frac{I(-s^2) + I(s^2)}{2}, \end{aligned}$$

using the double-angle formula $2\cos^2(\theta) = 1 + \cos(2\theta)$ and a change of variables $\theta \mapsto 2\theta$. Then, since $I(s) = I(-s)$, we have

$$I(s) = \frac{1}{2} I(s^2), \qquad \forall s \in \mathbb{R}.$$

Repeating this argument, we have

$$I(s) = \frac{1}{2^n} I\left(s^{2^n}\right), \qquad \forall n \in \mathbb{N}_0, \quad \forall s \in \mathbb{R}.$$

Furthermore, we have $I(0) = I(1) = 0$ by direct calculation. It then follows that

$$I(s) = \lim_{n\to\infty} \frac{1}{2^n} I\left(s^{2^n}\right) = 0, \qquad \forall s \in (0,1),$$

and so $I(s) = 0$ for $s \in [0,1]$. If instead $s > 1$, then $\frac{1}{s} \in (0,1)$ and so $I\left(\frac{1}{s}\right) = 0$. Thus,

$$\begin{aligned} 0 = I\left(\frac{1}{s}\right) &= \int_0^\pi \log\left(1 + \frac{2}{s}\cos(\theta) + \frac{1}{s^2}\right) \mathrm{d}\theta \\ &= \int_0^\pi \log\left(1 + 2s\cos(\theta) + s^2\right) - \log\left(s^2\right) \mathrm{d}\theta \\ &= I(s) - 2\pi\log(s) \quad \Rightarrow \quad I(s) = 2\pi\log(s), \qquad \forall s > 1. \end{aligned}$$

Hence, we have

$$I(s) = \begin{cases} 0, & s \in [0,1], \\ 2\pi\log(s), & s > 1. \end{cases}$$

Then, using the fact that

$$\mathsf{p}_{B_r(te_1)} = \frac{t^2}{\pi r^2} \int_0^{r/t} I(s) s \, \mathrm{d}s + \log(t),$$

we find that

$$\mathsf{p}_{B_r(te_1)} = \begin{cases} \log(t), & r \leqslant t, \\ \log(r) + \frac{1}{2}\left(\left(\frac{t}{r}\right)^2 - 1\right), & r > t. \end{cases}$$

Note that the case of $r \leqslant t$ could also be obtained by using the mean-value property of harmonic functions as $\Delta \mathsf{p} = 0$ in this region.

To calculate $[\mathsf{p}]_{\mathrm{BMO}}$ we need to evaluate the integral

$$⨍_{B_r(te_1)} \left|\mathsf{p}(x) - \mathsf{p}_{B_r(te_1)}\right| \mathrm{d}x = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \left|\mathsf{p}(te_1 + \tilde{r} e_r(\theta)) - \mathsf{p}_{B_r(te_1)}\right| \tilde{r} \, \mathrm{d}\theta \, \mathrm{d}\tilde{r},$$

in the region $t + r \leqslant 1$. We shall partition this region into two triangles,

$$\begin{aligned} T^- &= \left\{(t,r) \in \mathbb{R}^2 : t + r \leqslant 1, \ 0 \leqslant r \leqslant t\right\}, \\ T^+ &= \left\{(t,r) \in \mathbb{R}^2 : t + r \leqslant 1, \ 0 \leqslant t < r\right\}. \end{aligned}$$

We then consider each region $T^{\pm}$ separately.

- Let $(t, r) \in T^-$. Then

$$\fint_{B_r(te_1)} \left|\mathsf{p}(x) - \mathsf{p}_{B_r(te_1)}\right| \, \mathrm{d}x = \frac{1}{2\pi r^2} \int_0^r \int_0^{2\pi} \left|\log\left(\frac{t^2 + 2t\tilde{r}\cos(\theta) + \tilde{r}^2}{t^2}\right)\right| \tilde{r} \, \mathrm{d}\theta \, \mathrm{d}\tilde{r}$$
$$= \frac{1}{2\pi\alpha^2} \int_0^\alpha \int_0^{2\pi} \left|\log\left(1 + 2s\cos(\theta) + s^2\right)\right| s \, \mathrm{d}\theta \, \mathrm{d}s,$$

using the change of variables $s = \frac{\tilde{r}}{t}$ and defining $\alpha := \frac{r}{t} \leqslant 1$.

- Let $(t, r) \in T^+$. Then

$$\fint_{B_r(te_1)} \left|\mathsf{p}(x) - \mathsf{p}_{B_r(te_1)}\right| \, \mathrm{d}x$$
$$= \frac{1}{2\pi r^2} \int_0^r \int_0^{2\pi} \left|\log\left(\frac{t^2 + 2t\tilde{r}\cos(\theta) + \tilde{r}^2}{t^2}\right) + 2\log\left(\frac{t}{r}\right) - \left(\frac{t}{r}\right)^2 + 1\right| \tilde{r} \, \mathrm{d}\theta \, \mathrm{d}\tilde{r}$$
$$= \frac{1}{2\pi\alpha^2} \int_0^\alpha \int_0^{2\pi} \left|\log\left(1 + 2s\cos(\theta) + s^2\right) + \log\left(\alpha^{-2}\right) - \alpha^{-2} + 1\right| s \, \mathrm{d}\theta \, \mathrm{d}s,$$

using the same change of variables $s = \frac{\tilde{r}}{t}$ but now with $\alpha > 1$.

If we now define $h : [0, \infty) \to \mathbb{R}$ by

$$h(z) = \begin{cases} 0, & z \leqslant 1, \\ \log\left(z^{-2}\right) - z^{-2} + 1, & z > 1, \end{cases}$$

then we have

$$[\mathsf{p}]_{\mathrm{BMO}} = \sup_{\alpha \geqslant 0} \frac{1}{2\pi\alpha^2} \int_0^\alpha \int_0^{2\pi} \left|\log\left(1 + 2s\cos(\theta) + s^2\right) + h(\alpha)\right| s \, \mathrm{d}\theta \, \mathrm{d}s.$$

The supremum is attained at $\alpha = 1$ (where $t = r$ and we are at the interface of $T^\pm$) and so

$$[\mathsf{p}]_{\mathrm{BMO}} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \left|\log\left(1 + 2s\cos(\theta) + s^2\right)\right| s \, \mathrm{d}\theta \, \mathrm{d}s$$
$$= \frac{1}{\pi} \int_0^1 \int_0^{\pi} \left|\log\left(1 + 2s\cos(\theta) + s^2\right)\right| s \, \mathrm{d}\theta \, \mathrm{d}s.$$

Now, let $\bar{\theta}(s) \in [0, \pi]$ be the unique solution to $\cos\left(\bar{\theta}(s)\right) = -\frac{s}{2}$ for $s \in [-2, +2]$. We can then write

$$[\mathsf{p}]_{\mathrm{BMO}} = \frac{1}{\pi} \int_0^1 \int_0^{\bar{\theta}(s)} \log\left(1 + 2s\cos(\theta) + s^2\right) s \, \mathrm{d}\theta \, \mathrm{d}s$$
$$- \frac{1}{\pi} \int_0^1 \int_{\bar{\theta}(s)}^{\pi} \log\left(1 + 2s\cos(\theta) + s^2\right) s \, \mathrm{d}\theta \, \mathrm{d}s$$
$$= \frac{1}{\pi} \int_0^1 s\tilde{I}_1(s) - s\tilde{I}_2(s) \, \mathrm{d}s,$$

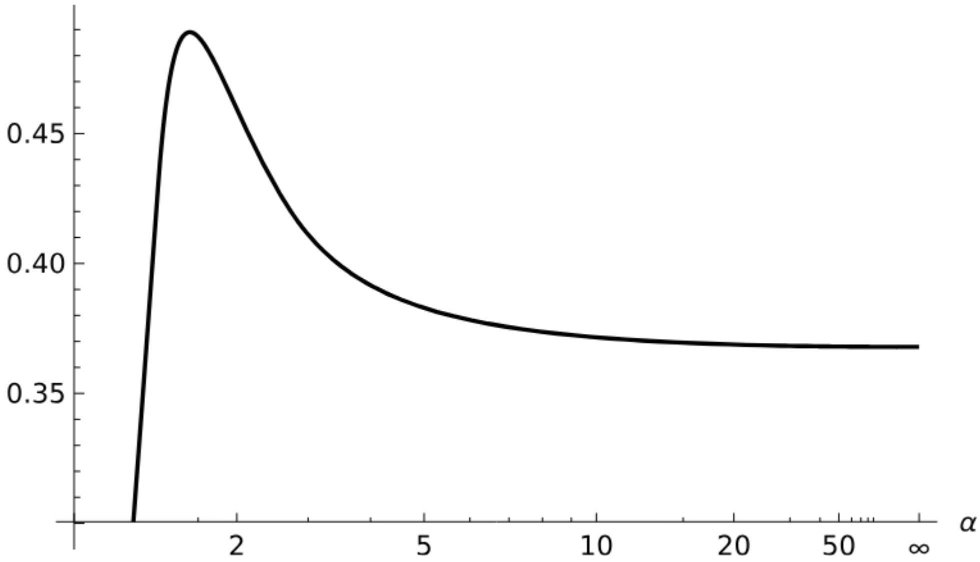

Figure 3.12: Plot showing the required integral for varying values of the parameter $\alpha \geqslant 0$.

where

$$\tilde{I}_1(s) := \int_0^{\bar{\theta}(s)} \log\left(1 + 2s\cos(\theta) + s^2\right) \mathrm{d}\theta,$$
$$\tilde{I}_2(s) := \int_{\bar{\theta}(s)}^{\pi} \log\left(1 + 2s\cos(\theta) + s^2\right) \mathrm{d}\theta.$$

Integrating by parts, we find that

$$[\mathrm{p}]_{\mathrm{BMO}} = \frac{1}{2\pi}\left[s^2\tilde{I}_1(s) - s^2\tilde{I}_2(s)\right]_0^1 - \frac{1}{2\pi}\int_0^1 s^2\tilde{I}_1'(s) - s^2\tilde{I}_2'(s)\,\mathrm{d}s.$$

Then, since

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\log(2 + 2\cos(\theta))\,\theta - 2\log\left(1 + e^{i\theta}\right) + \left(\frac{1}{2}\theta^2 + 2\mathrm{Li}_2\left(-e^{i\theta}\right)\right) i\right) = \log(2 + 2\cos(\theta)),$$

we have

$$\tilde{I}_1(0) = 0, \qquad \tilde{I}_1(1) = +\iota,$$
$$\tilde{I}_2(0) = 0, \qquad \tilde{I}_2(1) = -\iota,$$

where $\iota := 2\,\Im(\mathrm{Li}_2(\omega)) > 0$ and $\omega = e^{\frac{\pi i}{3}}$. Here $\mathrm{Li}_2(\cdot)$ denotes the dilogarithm [12]. The derivatives are given by

$$\tilde{I}_1'(s) = \int_0^{\bar{\theta}(s)} \frac{2s + 2\cos(\theta)}{1 + 2s\cos(\theta) + s^2}\,\mathrm{d}\theta,$$
$$\tilde{I}_2'(s) = \int_{\bar{\theta}(s)}^{\pi} \frac{2s + 2\cos(\theta)}{1 + 2s\cos(\theta) + s^2}\,\mathrm{d}\theta.$$

Note that, due to the definition of $\bar{\theta}$, the endpoint terms vanish. Focussing on $\tilde{I}_1'$, we have

$$\begin{aligned}
\tilde{I}_1'(s) &= \frac{1}{s}\int_0^{\bar{\theta}(s)} \frac{1+2s\cos(\theta)+s^2+(s^2-1)}{1+2s\cos(\theta)+s^2}\,\mathrm{d}\theta \\
&= \frac{\bar{\theta}(s)}{s} + \frac{1}{s}\int_0^{\bar{\theta}(s)} \frac{s^2-1}{1+2s\cos(\theta)+s^2}\,\mathrm{d}\theta \\
&= \frac{\bar{\theta}(s)}{s} + \frac{1}{s}\int_0^{\sqrt{\frac{2+s}{2-s}}} \frac{(s^2-1)(1+t^2)}{(1+s^2)(1+t^2)+2s(1-t^2)}\frac{2}{1+t^2}\,\mathrm{d}t \\
&= \frac{\bar{\theta}(s)}{s} + \frac{2}{s}\int_0^{\sqrt{\frac{2+s}{2-s}}} \frac{s^2-1}{(s+1)^2+(s-1)^2t^2}\,\mathrm{d}t \\
&= \frac{\bar{\theta}(s)}{s} + \frac{2}{s}\left[\tan^{-1}\left(\frac{s-1}{s+1}t\right)\right]_0^{\sqrt{\frac{2+s}{2-s}}} = \frac{\bar{\theta}(s)}{s} + \frac{2}{s}\tan^{-1}\left(\frac{s-1}{s+1}\sqrt{\frac{2+s}{2-s}}\right).
\end{aligned}$$

Similarly, we find that

$$\tilde{I}_2'(s) = \frac{\pi-\bar{\theta}(s)}{s} + \frac{2}{s}\left[\tan^{-1}\left(\frac{s-1}{s+1}\tan\left(\frac{\theta}{2}\right)\right)\right]_{\bar{\theta}(s)}^{\pi^-} = -\tilde{I}_1'(s),$$

for almost every $s \in [0,1]$. Now using the fact that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{8}s\sqrt{4-s^2} + \frac{1}{2}s^2\tan^{-1}\left(\frac{s-1}{s+1}\sqrt{\frac{2+s}{2-s}}\right) - \frac{3}{2}\tan^{-1}\left(\frac{s}{\sqrt{4-s^2}}\right)\right) \\
&\quad = \tan^{-1}\left(\frac{s-1}{s+1}\sqrt{\frac{2+s}{2-s}}\right),
\end{aligned}$$

we calculate

$$\begin{aligned}
[\mathsf{p}]_{\mathrm{BMO}} &= \frac{\iota}{\pi} - \frac{1}{\pi}\int_0^1 \bar{\theta}(s)s\,\mathrm{d}s - \frac{2}{\pi}\int_0^1 \tan^{-1}\left(\frac{s-1}{s+1}\sqrt{\frac{2+s}{2-s}}\right)s\,\mathrm{d}s \\
&= \frac{\iota}{\pi} - \frac{1}{\pi}\left(\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right) - \frac{2}{\pi}\left(\frac{3}{8}\sqrt{3}-\frac{\pi}{4}\right) = \frac{\iota-\sqrt{3}}{\pi} + \frac{1}{3} \approx 0.428136.
\end{aligned}$$

Thus, a sufficient condition for $\mathbb{E}_{\lambda\mathsf{p}} \geqslant 0$ is

$$\lambda \leqslant \lambda_* := \frac{C}{[\mathsf{p}]_{\mathrm{BMO}}} \approx 2.56345 \times 10^{-17}.$$

3.3.3. Linear Pressure Functions

For an example of a non-radially symmetric pressure function, we shall return to the linear example, given by

$$\mathsf{p}_\lambda(x) = \lambda\cdot x \qquad x \in B \subset \mathbb{R}^2,$$

with parameter $\lambda \in \mathbb{R}^2$. Since this pressure function is harmonic everywhere, we can use the mean value property to simplify calculation of $[\mathsf{p}_\lambda]_{\mathrm{BMO}}$. We have

$$\begin{aligned}
[\mathsf{p}_\lambda]_{\mathrm{BMO}} &= \sup_{B_r(x_0)\subset B} \fint_{B_r(x_0)} |\mathsf{p}_\lambda(x) - \mathsf{p}_\lambda(x_0)| \, \mathrm{d}x \\
&= \sup_{B_r(x_0)\subset B} \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r |\lambda \cdot (x_0 + \rho e_r(\theta)) - \lambda \cdot x_0| \, \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \\
&= \sup_{r\in[0,1]} \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r |\lambda \cdot e_r(\theta)| \, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\theta \\
&= \sup_{r\in[0,1]} \frac{r}{3\pi} \int_0^{2\pi} |\lambda| \cdot |\sin(\theta + \arg\lambda)| \, \mathrm{d}\theta \\
&= \frac{|\lambda|}{3\pi} \int_0^{2\pi} |\sin(\theta)| \, \mathrm{d}\theta \\
&= \frac{4}{3\pi} |\lambda| \,,
\end{aligned}$$

making use of the substitution $\theta \mapsto \theta - \arg\lambda$. Hence, a sufficient condition for $\mathbb{E}_{\mathsf{p}_\lambda} \geqslant 0$ is

$$|\lambda| \leqslant \lambda_* := \frac{3\pi C}{4} \approx 2.58594 \times 10^{-17}.$$

3.3.4. Non-Differentiable Pressure Functions

One of the main advantages of deriving sufficient conditions for non-negative excess using Theorem 3.1, as opposed to other estimates, is that it can be applied to pressure functions without restrictions on the derivative. As an extreme example, we consider the Weierstrass function [25]

$$W_{a,b}(t) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi t) \,, \qquad t \in \mathbb{R},$$

with parameters $a \in (0,1)$ and $b \in \mathbb{N}$ odd satisfying

$$ab > 1 + \frac{3\pi}{2}.$$

We then define a pressure function $\mathsf{p} : B \to \mathbb{R}$

$$\mathsf{p}(x) = W_{a,b}(|x|), \qquad x \in B \subset \mathbb{R}^2.$$

This pressure function is bounded but non-differentiable almost everywhere in B [25], making it impossible to apply estimates that require bounds on the weak derivative of p. However, we can again use the L^∞ estimate

$$[\mathsf{p}]_{\mathrm{BMO}} \leqslant 2 \, \|\mathsf{p}\|_\infty = \frac{2}{1-a}.$$

4

Necessary Conditions for a Non-Negative Excess

In the previous chapters, we have explored a variety of techniques for deriving sufficient conditions on the pressure function p so that $\mathbb{E}_{\mathsf{p}} \geqslant 0$. In this chapter, we shall instead turn our attention to deriving necessary conditions on p. This will gives us an idea of when excess functionals can be used to solve constrained optimisation problems and when we must seek out other methods. To obtain these necessary conditions we will consider $\mathbb{E}_{\mathsf{p}}(\varphi)$ for a specific φ, one that is inspired by the sharpness of Hadamard's inequality.

Proposition 4.1. *There exists a partition $B^{\pm}$ of B such that*

$$\mathbb{E}_{\mathsf{p}} \geqslant 0 \quad \Rightarrow \quad \left| \fint_{B^+} \mathsf{p} \, \mathrm{d}x - \fint_{B^-} \mathsf{p} \, \mathrm{d}x \right| \leqslant 2.$$

Proof. Consider $\varphi \in H_0^1(B; \mathbb{R}^2)$ such that $\nabla\varphi \in \mathrm{O}(2)$ almost everywhere (we shall assume existence of such a map for now). Then we define

$$B^{\pm} := \{x \in B : \det \nabla\varphi(x) = \pm 1\}. \tag{4.1}$$

Since $\varphi \mapsto \det \nabla\varphi$ is a null Lagrangian, we observe that

$$0 = \int_B \det \nabla\varphi|_{\partial B} \, \mathrm{d}x = \int_B \det \nabla\varphi \, \mathrm{d}x = \int_{B^+} \mathrm{d}x - \int_{B^-} \mathrm{d}x = \left|B^+\right| - \left|B^-\right|,$$

and so $|B^{\pm}| = \frac{1}{2}|B|$. Now we calculate

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \int_B 1 + \mathsf{p}(\chi_{B^+} - \chi_{B^-}) \, \mathrm{d}x = |B| \left(1 + \frac{1}{2}\fint_{B^+} \mathsf{p} \, \mathrm{d}x - \frac{1}{2}\fint_{B^-} \mathsf{p} \, \mathrm{d}x\right).$$

Thus, we require

$$\fint_{B^+} \mathsf{p} \, \mathrm{d}x - \fint_{B^-} \mathsf{p} \, \mathrm{d}x \geqslant -2.$$

Now we define $I^{\pm} := \mathrm{diag}(1, \pm 1)$. Repeating these calculations but replacing φ with $I^-\varphi$ yields the same results but with the roles of the $B^{\pm}$ interchanged. Hence, we obtain the necessary condition given in the statement. □

This inequality has previously been used in the context of piecewise constant pressure functions [10]. We will later apply this technique to some one-parameter family of pressure functions as an example. With this in mind, it will be useful to see how scaling the pressure function affects this condition.

Corollary 4.2. *Let $B^{\pm}$ be as in Proposition 4.1. Let* $\mathsf{p} : B \to \mathbb{R}$ *satisfy*

$$\kappa := \left| \fint_{B^+} \mathsf{p}\, \mathrm{d}x - \fint_{B^-} \mathsf{p}\, \mathrm{d}x \right| .$$

Then

$$\mathbb{E}_{\lambda \mathsf{p}} \geqslant 0 \quad \Rightarrow \quad |\lambda| \leqslant \frac{2}{\kappa}.$$

In the case when $\kappa = 0$, the necessary condition is trivial.

Proof. Substituting $\mathsf{p} \mapsto \lambda \mathsf{p}$ into the necessary condition in Proposition 4.1, we obtain

$$\left| \fint_{B^+} \lambda \mathsf{p}\, \mathrm{d}x - \fint_{B^-} \lambda \mathsf{p}\, \mathrm{d}x \right| \leqslant 2 \quad \Longleftrightarrow \quad |\lambda|\, \kappa \leqslant 2.$$

□

To proceed, we will need to establish the existence of a $\varphi \in H^1_0(B; \mathbb{R}^2)$ satisfying $\nabla \varphi \in \mathrm{O}(2)$ as used in the proof of Proposition 4.1.

4.1. Potential Wells and Partial Differential Inclusions

To make use of Proposition 4.1, we require a φ solving the partial differential inclusion

$$\begin{cases} \nabla \varphi(x) \in \mathrm{O}(2), & \text{a.e. } x \in B, \\ \varphi(x) = 0, & x \in \partial B. \end{cases} \tag{4.2}$$

This PDI is a specific example of a more general PDI, namely, the problem of potential wells, given by

$$\begin{cases} \nabla \varphi(x) \in \bigcup_{i=1}^{n} A_i\, \mathrm{SO}(2), & \text{a.e. } x \in \Omega, \\ \varphi(x) = \varphi_0(x), & x \in \partial \Omega. \end{cases} \tag{4.3}$$

where $A_1, \ldots, A_n$ are 2×2 matrices and the $A_i\, \mathrm{SO}(2)$ are referred to as wells. In (4.2), we have just two wells, $\mathrm{SO}^+(2) := \mathrm{SO}(2)$ and $\mathrm{SO}^-(2) := I^-\, \mathrm{SO}(2)$. The existence of solutions to (4.2) has been established [18, 40] not just for the domain $B \subset \mathbb{R}^2$, but for any open domain $\Omega \subset \mathbb{R}^n$.

Theorem 4.3. *[18, 7.28] Let $\varphi_0 : \bar{\Omega} \to \mathbb{R}^n$ be a piecewise C^1 function satisfying*

$$\| |\nabla \varphi_0|_2 \|_\infty < 1.$$

Then there exists a dense set of solutions $\varphi \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ *to*

$$\begin{cases} \nabla\varphi(x) \in \mathrm{O}(2), & \text{a.e. } x \in \Omega, \\ \varphi(x) = \varphi_0(x), & x \in \partial\Omega. \end{cases}$$

Clearly $\varphi_0 \equiv 0$ satisfies the requirements for this theorem and so we have the existence of solutions to (4.2) even if we change the domain. The ability to solve on an arbitrary domain is of significance as it may be easier to construct an explicit solution on some domains when than others. If, for example, we can construct a solution on a square domain $Q = [-2,+2]^2$, this will be enough to derive a necessary condition for non-negativity of $\mathbb{E}_\mathsf{p}$.

Proposition 4.4. *Let* $\varphi^* : Q \to \mathbb{R}^2$ *solve*

$$\begin{cases} \nabla\varphi^*(x) \in \mathrm{O}(2), & \text{a.e. } x \in Q, \\ \varphi^*(x) = 0, & x \in \partial Q, \end{cases} \tag{4.4}$$

where $Q := [-2,+2]^2$. *Let* $\{Q_i\}_{i=1}^m$ *be a collection of pairwise disjoint square subsets of* B *with centres* x_i *and side lengths* ℓ_i, *respectively and define* $\mathsf{P} : Q \to \mathbb{R}$ *by*

$$\mathsf{P}(y) = \sum_{i=1}^m \mathsf{p}\left(x_i + \frac{\ell_i}{4}y\right).$$

Then

$$\mathbb{E}_\mathsf{p}(\varphi) \geqslant 0 \quad \Rightarrow \quad \left| \fint_{Q^+} \mathsf{P}(y)\,\mathrm{d}y - \fint_{Q^-} \mathsf{P}(y)\,\mathrm{d}y \right| \leqslant 2m. \tag{4.5}$$

Proof. We start by defining $\varphi : B \to \mathbb{R}^2$ by

$$\varphi(x) = \sum_{i=1}^m \varphi^*\left(\frac{4}{\ell_i}(x - x_i)\right)\chi_{Q_i}(x).$$

Due to the boundary condition in (4.4), we conclude that φ is continuous. Then, using the PDI, we have

$$\begin{aligned} \mathbb{E}_\mathsf{p}(\varphi) &= \sum_{i=1}^m \left(\frac{4}{\ell_i}\right)^2 \int_{Q_i} \frac{1}{2}\left|\nabla\varphi^*\left(\frac{4}{\ell_i}(x-x_i)\right)\right|^2 + \mathsf{p}(x)\det\nabla\varphi^*\left(\frac{4}{\ell_i}(x-x_i)\right)\mathrm{d}x \\ &= \sum_{i=1}^m \int_Q \frac{1}{2}|\nabla\varphi^*(y)|^2 + \mathsf{p}\left(x_i + \frac{\ell_i}{4}y\right)\det\nabla\varphi^*(y)\,\mathrm{d}y \\ &= \sum_{i=1}^m \int_Q 1 + \mathsf{p}\left(x_i + \frac{\ell_i}{4}y\right)(\chi_{Q^+}(y) - \chi_{Q^-}(y))\,\mathrm{d}y \\ &= |Q|\left(m + \frac{1}{2}\fint_{Q^+}\mathsf{P}(y)\,\mathrm{d}y - \frac{1}{2}\fint_{Q^-}\mathsf{P}(y)\,\mathrm{d}y\right), \end{aligned}$$

with a change of variables given by $y = x_i + \frac{\ell_i}{4}$ on each of the Q_i. Here we have used a similar notation to that of (4.1), noting that $|Q^{\pm}| = \frac{1}{2}|Q| = 8$. Thus, we require that

$$\fint_{Q^+} \mathrm{P}(y)\,\mathrm{d}y - \fint_{Q^-} \mathrm{P}(y)\,\mathrm{d}y \geqslant -2m.$$

We can get the other side of inequality (4.5) by swapping φ^* with $I^-\varphi^*$ which interchanges the role of Q^+ with Q^-. □

Note that the right-hand side of the necessary condition is proportional to m, which suggests that we may get improvements in the bound by using fewer squares (small m). We can also derive an analogous version of Corollary 4.2.

Corollary 4.5. *Let $Q^{\pm}$ and P be as in Proposition 4.4 with P satisfying*

$$\kappa := \left| \fint_{Q^+} \mathrm{P}(y)\,\mathrm{d}y - \fint_{Q^-} \mathrm{P}(y)\,\mathrm{d}y \right|,$$

Then

$$\mathbb{E}_{\lambda \mathrm{p}} \geqslant 0 \quad \Rightarrow \quad |\lambda| \leqslant \frac{2m}{\kappa}$$

In the case when $\kappa = 0$, the necessary condition is trivial.

Proof. The proof is almost identical to that of Corollary 4.2, the on;y difference being that Proposition 4.4 is used instead of Proposition 4.1. □

It remains to construct an explicit solution to (4.4).

4.2. Constructing Measure Preserving Maps

We now seek to construct an explicit solution φ^* to (4.4). The case of a three dimensional square, more generally any bounded set in $\mathbb{R}^3$, has already been covered [14] and so we shall try to adapt this to work in two dimensions.

Theorem 4.6. *[14, 3.1] Let $\Omega \subset \mathbb{R}^3$ be an open, bounded domain. Then there exists a Lipschitz continuous map $\varphi : \Omega \to \mathbb{R}^3$, with Lipschitz constant one, such that*

$$\begin{cases} \nabla\varphi(x) \in \mathcal{R} \subset \mathrm{SO}^+(3) \cup \mathrm{SO}^-(3), & \text{a.e. } x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Given an $x = (x_1, x_2) \in \mathbb{R}^2$, we define the functions $|X_i|(x), |X_s|(x)$ to be the smallest and largest values in $(|x_1|, |x_2|)$ respectively. These are continuous functions that are invariant under permutation and taking absolute values of the entries of x.

Next, we define the functions $i, s : \mathbb{R}^2 \to \{1, 2\}$ to give the position of the smallest and largest values in $(|x_1|, |x_2|)$ respectively. Note that these functions are locally constant.

Now define $f_k : \mathbb{R} \to \mathbb{R}$ (for $k \in \mathbb{Z}$) by taking f_1 to be the 1-periodic extension of

$$t \mapsto \min\{t, 1-t\}, \quad t \in [0, 1],$$

and

$$f_k(t) = \frac{1}{2^{k-1}} f_1\left(2^{k-1}t\right).$$

We observe that these are all even continuous functions. We then define a function $\varphi^1 : \frac{1}{2}\bar{Q} \to \mathbb{R}^2$ by

$$\varphi_1^1(x) = \min\{f_1(|X_i|(x)), f_1(|X_s|(x))\},$$

$$\varphi_2^1(x) = \begin{cases} f_2(|X_s|(x)), & |X_i|(x) + |X_s|(x) \leqslant 1, \\ f_2(|X_i|(x)), & |X_i|(x) + |X_s|(x) \geqslant 1. \end{cases}$$

One can verify that φ^1 is Lipschitz on $\frac{1}{2}Q$. We now establish a key characteristic of φ^1 that will be needed for the construction.

Lemma 4.7. *For all* $x_1 \in (-1, +1)$*, we have* $\varphi^1(x_1, 1) = \varphi^1(x_1 \bmod 1, 1)$.

Proof. Taking $x = (x_1, 1)$ with $x_1 \in (-1, +1)$, we have

$$|X_i|(x), = |x_1|,$$

$$|X_s|(x), = 1.$$

Thus we obtain

$$\varphi_1^1(x_1, 1) = \min\{f_1(|x_1|), f_1(1)\},$$

$$\varphi_2^1(x_1, 1) = \begin{cases} f_2(1) & |x_1| = 0, \\ f_2(|x_1|) & |x_1| > 0. \end{cases}$$

Now we use the that f_1, f_2 are non-negative, even and satisfy

$$f_1(1) = f_2(1) = 0.$$

Hence

$$\varphi_1^1(x_1, 1) = \min\{f_1(|x_1|), 0\},$$

$$\varphi_2^1(x_1, 1) = \begin{cases} 0, & |x_1| = 0, \\ f_2(|x_1|), & |x_1| > 0. \end{cases}$$

Finally, we have

$$\begin{aligned}\varphi_1^1(x_1, 1) &= 0,\\ \varphi_2^1(x_1, 1) &= f_2(x_1).\end{aligned}$$

Clearly $\varphi^1(x_1, 1) = f_2(x_1)e_2$ is 1-periodic (in fact $\frac{1}{2}$-periodic) in x_1. □

We also calculate

$$\sup\left\{\left|\varphi_j^1(x)\right| : x \in \frac{1}{2}\bar{Q},\ j \in \{1,2\}\right\} = \frac{1}{2},$$

and

$$\nabla\varphi_1^1(x) = \begin{cases} +\operatorname{sgn}(x_{i(x)})e_{i(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| < 1,\\ -\operatorname{sgn}(x_{s(x)})e_{s(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| > 1,\end{cases}$$

$$\nabla\varphi_2^1(x) = \begin{cases} f_2'(x_{s(x)})e_{s(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| < 1,\\ f_2'(x_{i(x)})e_{i(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| > 1.\end{cases}$$

Since $\left|f_2'(t)\right| = 1$ for $t \notin \frac{1}{4}\mathbb{Z}$, we have $\nabla\varphi^1(x) \in O(2)$ almost everywhere. We now define the function $\psi : \frac{1}{4}\bar{Q} \to \mathbb{R}$ by

$$\psi_1(x) = \begin{cases} f_3(|X_s|(x)), & |X_i|(x) + |X_s|(x) \leqslant \frac{1}{2},\\ f_3(|X_i|(x)), & |X_i|(x) + |X_s|(x) \geqslant \frac{1}{2},\end{cases}$$

$$\psi_2(x) = \min\{f_2(|X_i|(x)), f_2(|X_s|(x))\},$$

for $x \in Q_s$, and

$$\psi_1(x) = \begin{cases} f_3(|X_i|(x)), & |X_i|(x) + |X_s|(x) \leqslant \frac{1}{2},\\ f_3(|X_s|(x)), & |X_i|(x) + |X_s|(x) \geqslant \frac{1}{2},\end{cases}$$

$$\psi_2(x) = \max\{f_2(|X_i|(x)), f_2(|X_s|(x))\},$$

for $x \in Q_i$, where Q_i, Q_s are the sets for which x_2 is the smallest and largest in absolute value in $\{x_1, x_2\}$ respectively.

$$\begin{aligned} Q_i &:= \{(x_1, x_2) : |x_2| \leqslant |x_1|\},\\ Q_s &:= \{(x_1, x_2) : |x_2| \geqslant |x_1|\}.\end{aligned} \tag{4.6}$$

Again, one can verify that ψ is Lipschitz on $\frac{1}{4}Q$. Furthermore, ψ posses a number of other useful properties.

Proposition 4.8. *The function ψ has the following properties:*

1. $\psi(x_1, x_2) = \psi(|x_1|, |x_2|)$.
2. $\psi(x_1, x_2) = \psi(x_2, x_1)$.

Proof. This follows directly from the properties of $|X_i|$ and $|X_s|$. □

Now we calculate

$$\sup\left\{\left|\psi_j(x)\right| : x \in \frac{1}{4}\bar{Q},\ j \in \{1,2\}\right\} = \frac{1}{2},$$

and

$$\nabla\psi_1(x) = \begin{cases} f_3'(x_{s(x)})e_{s(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| < \frac{1}{2}, \\ f_3'(x_{i(x)})e_{i(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| > \frac{1}{2}, \end{cases}$$

$$\nabla\psi_2(x) = \begin{cases} f_2'(x_{i(x)})e_{i(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| < \frac{1}{2}, \\ f_2'(x_{s(x)})e_{s(x)}, & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| > \frac{1}{2}, \end{cases}$$

for $x \in Q_s$, and

$$\nabla\psi_1(x) = \begin{cases} f_3'(x_{i(x)})e_{i(x)} & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| < \frac{1}{2}, \\ f_3'(x_{s(x)})e_{s(x)} & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| > \frac{1}{2}, \end{cases}$$

$$\nabla\psi_3(x) = \begin{cases} f_2'(x_{s(x)})e_{s(x)} & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| < \frac{1}{2}, \\ f_2'(x_{i(x)})e_{i(x)} & \left|x_{i(x)}\right| + \left|x_{s(x)}\right| > \frac{1}{2}, \end{cases}$$

for $x \in Q_i$, so $\nabla\psi(x) \in \mathrm{O}(2)$ almost everywhere in $\frac{1}{4}\bar{Q}$. We will need to establish some relationships between φ^1 and ψ that will ensure the resulting construction is continuous.

Lemma 4.9. *We have the following properties of φ^1 and ψ:*

1. $\varphi^1(x_1, 1) = \psi\left(x_1 \bmod 1 - \frac{1}{2}, 0\right)$.
2. *For all $\xi_1 \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, we have*

$$\psi_{r-1}(\xi_1, 0) = 2\psi_r\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\right),$$

where the indices are taken modulo 2.

Proof. For the first part, consider an arbitrary t. We then have

$$\psi_1(t, 0) = \begin{cases} f_3(|t|), & |t| \leqslant \frac{1}{2}, \\ f_3(0), & |t| \geqslant \frac{1}{2}, \end{cases}$$

$$\psi_2(t, 0) = \min\{f_2(0), f_2(|t|)\},$$

for $(t, 0) \in Q_s$, and

$$\psi_1(t, 0) = \begin{cases} f_3(0), & |t| \leqslant \frac{1}{2}, \\ f_3(|t|), & |t| \geqslant \frac{1}{2}, \end{cases}$$
$$\psi_2(t, 0) = \max\{f_2(0), f_2(|t|)\},$$

for $(t, 0) \in Q_i$. We also note that

$$f_2(0) = f_3(0) = 0.$$

Taking $t = x_1 \bmod 1 - \frac{1}{2} \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, then

$$(t, 0) \in Q_s^2 \iff t = 0,$$
$$(t, 0) \in Q_i^2 \iff t \neq 0,$$

and thus we obtain

$$\psi_1(t, 0) = f_3(t) = 0,$$
$$\psi_2(t, 0) = 0,$$

for $t = 0$, and

$$\psi_1(t, 0) = 0,$$
$$\psi_2(t, 0) = f_2(t),$$

for $t \neq 0$. This can be written more simply as just $\psi(t, 0) = f_2(t)e_2$. The proof then follows from the fact that f_2 is $\frac{1}{2}$-periodic and so

$$f_2(t) = f_2(x_1).$$

Moving on to (2), we have already determined that $\psi(\xi_1, 0) = f_2(\xi_1)e_2$, for any $\xi_1 \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$. Also, for any $x_1 \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, we have

$$\psi_1\left(x_1, \frac{1}{2}\right) = \begin{cases} f_3\left(\frac{1}{2}\right), & |x_1| = 0, \\ f_3(|x_1|), & |x_1| \geqslant 0, \end{cases}$$
$$\psi_2\left(x_1, \frac{1}{2}\right) = \min\left\{f_2(|x_1|), f_2\left(\frac{1}{2}\right)\right\},$$

for $\left(x_1, \frac{1}{2}\right) \in Q_s$, and

$$\psi_1\left(x_1, \frac{1}{2}\right) = \begin{cases} f_3(|x_1|), & |x_1| = 0, \\ f_3\left(\frac{1}{2}\right), & |x_1| \geqslant 0, \end{cases}$$
$$\psi_2\left(x_1, \frac{1}{2}\right) = \max\left\{f_2(|x_1|), f_2\left(\frac{1}{2}\right)\right\},$$

for $\left(x_1, \frac{1}{2}\right) \in Q_i$. We simplify in a similar manner to the previous part and find that $\psi\left(x_1, \frac{1}{2}\right) = f_3(x_1)e_1$. Now, taking $x_1 = \frac{1}{2}\xi_1 + \frac{1}{4}$, we have

$$\psi_1\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\right) = f_3\left(\frac{1}{2}\xi_1 + \frac{1}{4}\right) = \frac{1}{2}f_2(\xi_1) = \frac{1}{2}\psi_2(\xi_1, 0)$$

and $\psi_2\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\right) = 0 = \frac{1}{2}\psi_1(\xi_1, 0)$, completing the proof. □

We now define the *layering* function $\ell^1 : \mathbb{R} \times \left[-\frac{1}{2}, +\frac{1}{2}\right] \to \mathbb{R}^2$, by

$$\ell^1(x_1, x_2) = \psi\left(x_1 \mod 1 - \frac{1}{2}, x_2\right).$$

We immediately observe a symmetry of ℓ^1.

Lemma 4.10. *We have* $\ell^1(x_1, x_2) = \ell^1(|x_1|, |x_2|)$.

Proof. This follows directly from the properties of ψ and the fact that

$$\left|x_1 \mod 1 - \frac{1}{2}\right| = \left||x_1| \mod 1 - \frac{1}{2}\right|.$$

More specifically, we have

$$\begin{aligned}
\ell^1(x_1, x_2) &= \psi\left(x_1 \mod 1 - \frac{1}{2}, x_2\right) \\
&= \psi\left(\left|x_1 \mod 1 - \frac{1}{2}\right|, |x_2|\right) \\
&= \psi\left(\left||x_1| \mod 1 - \frac{1}{2}\right|, |x_2|\right) \\
&= \psi\left(|x_1| \mod 1 - \frac{1}{2}, |x_2|\right) \\
&= \ell^1(|x_1|, |x_2|).
\end{aligned}$$

□

Next, we define $\ell^n : \mathbb{R} \times \left[-\frac{1}{2^n}, +\frac{1}{2^n}\right] \to \mathbb{R}^2$, by

$$\ell^n(x_1, x_2) = \frac{1}{2^{n-1}}\ell^1\left(2^{n-1}x_1, 2^{n-1}x_2\right)$$

for each $n \in \mathbb{N}$. Then Lemma 4.10 extends to all the ℓ^n. We also obtain an important continuity result for the ℓ^n.

Lemma 4.11. *For each* $n \in \mathbb{N}$, *we have*

$$\ell^{n+1}_{r-1}(x_1, 0) = \ell^n_r\left(x_1, \frac{1}{2^n}\right),$$

where the indices are taken modulo 2.

Proof. We calculate

$$\begin{aligned}\ell^{n+1}(x_1,0) &= \frac{1}{2^n}\ell^1(2^n x_1, 0)\\ &= \frac{1}{2^n}\psi\left(2^n x_1 \mod 1 - \frac{1}{2}, 0\right)\\ &= \frac{1}{2^n}\mathcal{T}\left(2\psi\left(\frac{1}{2}\left(2^n x_1 \mod 1 - \frac{1}{2}\right) + \frac{1}{4}, \frac{1}{2}\right)\right),\end{aligned}$$

using Lemma 4.9 (2). Here $\mathcal{T}$ denotes the linear operator corresponding to the permutation $(1\,2)$, that is,

$$\mathcal{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}\ell^{n+1}(x_1,0) &= \mathcal{T}\left(\frac{1}{2^{n-1}}\psi\left(\frac{1}{2}\left(2^n x_1 \mod 1\right), \frac{1}{2}\right)\right)\\ &= \mathcal{T}\left(\frac{1}{2^{n-1}}\psi\left(2^{n-1}x_1 - \frac{1}{2}\lfloor 2^n x_1\rfloor, \frac{1}{2}\right)\right),\end{aligned}$$

Now, since

$$t \mapsto \psi\left(t, \frac{1}{2}\right) = f_3(t)e_1$$

is $\frac{1}{2}$-periodic, we can conclude that

$$\begin{aligned}\ell^{n+1}(x_1,0) &= \mathcal{T}\left(\frac{1}{2^{n-1}}\ell^1\left(2^{n-1}x_1, \frac{1}{2}\right)\right)\\ &= \mathcal{T}\ell^n\left(x_1, \frac{1}{2^n}\right).\end{aligned}$$

□

We now define the *layers* $\mathcal{L}^n$, for each $n \in \mathbb{N}$, by

$$\begin{aligned}\mathcal{L}^n &= \left\{(x_1,x_2) : \sum_{k=0}^{n-2}\frac{1}{2^k} \leqslant |x_2| \leqslant \sum_{k=0}^{n-1}\frac{1}{2^k} \text{ and } |x_1| \leqslant |x_2|\right\}\\ &= \left\{(x_1,x_2) : 2 - 2^{2-n} \leqslant |x_2| \leqslant 2 - 2^{1-n} \text{ and } |x_1| \leqslant |x_2|\right\}.\end{aligned}$$

We have already defined φ^1 on $\mathcal{L}^1$. We further define φ^n on $\mathcal{L}^n$ by

$$\varphi^n_j(x_1,x_2) = \ell^{n-1}_{j-(n+1)}\left(x_1, |x_2| - \sum_{k=0}^{n-2}\frac{1}{2^k}\right)$$

for each $n \geqslant 2$. The $\mathcal{L}^n$ only cover half the domain but we will be able to use some symmetry to extend the construction to the rest of the domain.

Proposition 4.12. *We have the following property of φ^n:*

$$\varphi^n(x_1, x_2) = \varphi^n(|x_1|, |x_2|).$$

Proof. If $(x_1, x_2) \in \mathcal{L}^n$, then $|x_2| - \sum_{k=0}^{n-2} \frac{1}{2^k} \geqslant 0$ so

$$|x_2| - \sum_{k=0}^{n-2} \frac{1}{2^k} = \left| |x_2| - \sum_{k=0}^{n-2} \frac{1}{2^k} \right|.$$

The result then follows from Lemma 4.10. □

We will need to establish continuity of the φ^n as we go from one layer to the next.

Lemma 4.13. *We have continuity from φ^{n-1} to φ^n, in the sense that*

$$\varphi_j^n\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \varphi_j^{n-1}\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) \qquad n \geqslant 2.$$

Proof. We first observe that

$$\varphi_j^n\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \ell_{j-(n+1)}^{n-1}\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k} - \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \ell_{j-(n+1)}^{n-1}(x_1, 0)$$

and

$$\varphi_j^{n-1}\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \ell_{j-n}^{n-2}\left(x_1, \frac{1}{2^{n-2}}\right).$$

The proof then follows from making the substitution

$$n \mapsto n-2,$$
$$r \mapsto j-n,$$

in Lemma 4.11. □

We then extend each φ^n to the (square) annulus

$$\begin{aligned}\mathcal{A}^n &:= \left\{x : \sum_{k=0}^{n-2} \frac{1}{2^k} \leqslant |x|_\infty \leqslant \sum_{k=0}^{n-1} \frac{1}{2^k}\right\} \\ &= \left\{x : 2 - 2^{2-n} \leqslant |x|_\infty \leqslant 2 - 2^{1-n}\right\},\end{aligned}$$

where $|x|_\infty := \min\{|x_1|, |x_2|\}$, by setting

$$\varphi^n(x) = \varphi^n(|X_i|(x), |X_s|(x)),$$

which will not break continuity. We also calculate

$$\sup\left\{\left|\varphi_j^n(x)\right| : x \in \mathcal{A}^n\right\} \leqslant \frac{1}{2^{n-1}}$$

To compute the gradient $\nabla\varphi^n$ at almost any point $x = \bar{x}$ (excluding the null set formed by the boundaries joining the various pieces), we recall that there exists integers

$$\bar{\imath} = i(\bar{x}), \qquad \bar{s} = s(\bar{x}),$$

such that $\varphi^n(x) = \varphi^n(x_{\bar{\imath}}, x_{\bar{s}})$ for all x in a neighbourhood of $\bar{x}$. We then calculate the gradient with respect to $(x_{\bar{\imath}}, x_{\bar{s}})$ to be

$$\begin{aligned}
\nabla_{\bar{\imath},\bar{s}}\varphi_j^n(x_{\bar{\imath}}, x_{\bar{s}}) &= \nabla_{\bar{\imath},\bar{s}}\ell_{j-(n+1)}^{n-1}\left(x_{\bar{\imath}}, |x_{\bar{s}}| - \sum_{k=0}^{n-2}\frac{1}{2^k}\right) \\
&= \nabla_{\bar{\imath},\bar{s}}\frac{1}{2^{n-2}}\ell_{j-(n+1)}^{1}\left(2^{n-2}x_{\bar{\imath}}, 2^{n-2}\left(|x_{\bar{s}}| - \sum_{k=0}^{n-2}\frac{1}{2^k}\right)\right) \\
&= \nabla_{\bar{\imath},\bar{s}}\frac{1}{2^{n-2}}\psi_{j-(n+1)}\left(2^{n-2}x_{\bar{\imath}} \mod 1, 2^{n-2}\left(|x_{\bar{s}}| - \sum_{k=0}^{n-2}\frac{1}{2^k}\right)\right) \\
&= \nabla_{\bar{\imath},\bar{s}}\psi_{j-(n+1)}(\xi_{\bar{\imath}}, \xi_{\bar{s}}) \cdot \Sigma
\end{aligned}$$

almost everywhere, by the chain rule. Here

$$(\xi_{\bar{\imath}}, \xi_{\bar{s}}) = \left(2^{n-2}x_{\bar{\imath}} \mod 1, 2^{n-2}\left(|x_{\bar{s}}| - \sum_{k=0}^{n-2}\frac{1}{2^k}\right)\right), \qquad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(x_{\bar{s}}) \end{pmatrix}.$$

Then there is a permutation matrix P (depending only on n) such that

$$\nabla_{\bar{\imath},\bar{s}}\varphi^n(x_{\bar{\imath}}, x_{\bar{s}}) = \nabla_{\bar{\imath},\bar{s}}\psi(\xi_{\bar{\imath}}, \xi_{\bar{s}}) \cdot P\Sigma \in \mathrm{O}(2).$$

Finally, we define $\varphi^* : Q \to \mathbb{R}^2$ by

$$\varphi^*(x) = \sum_{n\in\mathbb{N}} \varphi^n(x)\chi_{\mathcal{A}^n}(x),$$

which is Lipschitz and satisfies all the desired properties.

Theorem 4.14. *The map φ^* solves 4.4.*

Proof. We have already established that φ^* satisfies the PDI. We have also shown that $|\varphi| \leqslant \frac{1}{2^{n-1}}$ on $\mathcal{A}^n$ and so $\varphi(x) \to 0$ as $x \to \partial Q$. □

Due to the reduction in dimension from three to two, there are a number of simplifications that can be made in this construction that will make calculations easier. Firstly, an equivalent form for $\varphi^1 : \frac{1}{2}\bar{Q} \to \mathbb{R}^2$ is given by

$$\varphi_1^1(x) = \min\{f_1(|x_1|), f_1(|x_2|)\},$$

$$\varphi_2^1(x) = \begin{cases} f_2(|X_s|(x)), & |x|_1 \leqslant 1, \\ f_2(|X_i|(x)), & |x|_1 \geqslant 1, \end{cases}$$

where $|x|_1 := |x_1| + |x_2|$. For $\psi : \frac{1}{4}\overline{Q} \to \mathbb{R}$, we have

$$\psi_1(x) = \begin{cases} f_3(|x_2|), & |x|_1 \leqslant \frac{1}{2}, \\ f_3(|x_1|), & |x|_1 \geqslant \frac{1}{2}, \end{cases}$$

$$\psi_2(x) = \begin{cases} \min\{f_2(|x_1|), f_2(|x_2|)\}, & |x_1| \leqslant |x_2|\,, \\ \max\{f_2(|x_1|), f_2(|x_2|)\}, & |x_1| \geqslant |x_2|\,. \end{cases}$$

We can expand the conditions for the pieces to get

$$\varphi_1^1(x) = \begin{cases} f_1(|x_1|), & f_1(|x_1|) \leqslant f_1(|x_2|), \\ f_1(|x_2|), & f_1(|x_1|) \geqslant f_1(|x_2|), \end{cases}$$

$$\varphi_2^1(x) = \begin{cases} f_2(|x_1|), & |x|_1 \leqslant 1 \text{ and } |x_1| \geqslant |x_2| \text{ or } |x|_1 \geqslant 1 \text{ and } |x_1| \leqslant |x_2|\,, \\ f_2(|x_2|), & |x|_1 \leqslant 1 \text{ and } |x_1| \leqslant |x_2| \text{ or } |x|_1 \geqslant 1 \text{ and } |x_1| \geqslant |x_2|\,, \end{cases}$$

$$\psi_1(x) = \begin{cases} f_3(|x_1|), & |x|_1 \geqslant \frac{1}{2}, \\ f_3(|x_2|), & |x|_1 \leqslant \frac{1}{2}, \end{cases}$$

$$\psi_2(x) = \begin{cases} f_2(|x_1|), & f_2(|x_1|) \leqslant f_2(|x_2|) \text{ and } |x_1| \leqslant |x_2| \text{ or } f_2(|x_1|) \geqslant f_2(|x_1|) \text{ and } |x_1| \geqslant |x_2|\,, \\ f_2(|x_2|), & f_2(|x_1|) \leqslant f_2(|x_2|) \text{ and } |x_1| \geqslant |x_2| \text{ or } f_2(|x_1|) \geqslant f_2(|x_1|) \text{ and } |x_1| \leqslant |x_2|\,. \end{cases}$$

Focussing on the conditions in these piecewise expressions, one can verify the following equivalences

$$\begin{aligned} f_1(|x_1|) \leqslant f_1(|x_2|) &\iff |x|_1 \leqslant 1 \text{ and } |x_1| \leqslant |x_2| \text{ or } |x|_1 \geqslant 1 \text{ and } |x_1| \geqslant |x_2|\,, \\ f_1(|x_1|) \geqslant f_1(|x_2|) &\iff |x|_1 \leqslant 1 \text{ and } |x_1| \geqslant |x_2| \text{ or } |x|_1 \geqslant 1 \text{ and } |x_1| \leqslant |x_2|\,, \\ |x|_1 \geqslant \frac{1}{2} &\iff f_2(|x_2|) \text{ and } |x_1| \geqslant |x_2| \text{ or } f_2(|x_1|) \geqslant f_2(|x_1|) \text{ and } |x_1| \leqslant |x_2|\,, \\ |x|_1 \leqslant \frac{1}{2} &\iff f_2(|x_2|) \text{ and } |x_1| \leqslant |x_2| \text{ or } f_2(|x_1|) \geqslant f_2(|x_1|) \text{ and } |x_1| \geqslant |x_2|\,. \end{aligned}$$

Since the f_k are all even functions, we can drop most of the absolute values and simplify as follows:

$$\varphi^1(x) = \begin{cases} (f_1(x_1), f_2(x_2))^{\mathsf{T}}\,, & |x|_1 \leqslant 1 \text{ and } |x_1| \leqslant |x_2| \text{ or } |x|_1 \geqslant 1 \text{ and } |x_1| \geqslant |x_2|\,, \\ (f_1(x_2), f_2(x_1))^{\mathsf{T}}\,, & |x|_1 \leqslant 1 \text{ and } |x_1| \geqslant |x_2| \text{ or } |x|_1 \geqslant 1 \text{ and } |x_1| \leqslant |x_2|\,, \end{cases}$$

$$\psi(x) = \begin{cases} (f_3(x_1), f_2(x_2))^{\mathsf{T}}\,, & |x|_1 \geqslant \frac{1}{2}, \\ (f_3(x_2), f_2(x_1))^{\mathsf{T}}\,, & |x|_1 \leqslant \frac{1}{2}. \end{cases}$$

If we now assume that $x \in \mathcal{L} := \bigcup_{n=1}^{\infty} \mathcal{L}^n$, we can simply write

$$\varphi^1(x) = \begin{cases} (f_1(x_1), f_2(x_2))^{\mathsf{T}}, & |x|_1 \leqslant 1, \\ (f_1(x_2), f_2(x_1))^{\mathsf{T}}, & |x_1| \geqslant 1, \end{cases}$$

$$\psi(x) = \begin{cases} (f_3(x_1), f_2(x_2))^{\mathsf{T}}, & |x|_1 \geqslant \frac{1}{2}, \\ (f_3(x_2), f_2(x_1))^{\mathsf{T}}, & |x|_1 \leqslant \frac{1}{2}. \end{cases}$$

We then calculate the gradients to be

$$\nabla\varphi^1(x) = \begin{cases} \begin{pmatrix} f_1'(x_1) & 0 \\ 0 & f_2'(x_2) \end{pmatrix}, & |x|_1 \leqslant 1, \\ \begin{pmatrix} 0 & f_1'(x_2) \\ f_2'(x_1) & 0 \end{pmatrix}, & |x|_1 \geqslant 1, \end{cases}$$

$$\nabla\psi(x) = \begin{cases} \begin{pmatrix} f_3'(x_1) & 0 \\ 0 & f_2'(x_2) \end{pmatrix}, & |x|_1 \geqslant \frac{1}{2}, \\ \begin{pmatrix} 0 & f_3'(x_2) \\ f_2'(x_1) & 0 \end{pmatrix}, & |x|_1 \leqslant \frac{1}{2}. \end{cases}$$

Since $\left|f_k'\right| = 1$ almost everywhere, the gradients are all in $O(2)$ almost everywhere, as expected. Recall that $\ell^n : \mathbb{R} \times \left[-\frac{1}{2^n}, +\frac{1}{2^n}\right] \to \mathbb{R}^2$ is defined by

$$\ell^1(x_1, x_2) = \psi\left(x_1 \bmod 1 - \frac{1}{2}, x_2\right)$$
$$\ell^n(x_1, x_2) = \frac{1}{2^{n-1}}\ell^1(2^{n-1}x_1, 2^{n-1}x_2)$$

and so $\varphi^n : \mathcal{L}^n \to \mathbb{R}^2$ (for $n > 1$) is given by

$$\varphi^n(x_1, x_2) = \mathcal{T}^{n-1}\ell^{n-1}\left(x_1, |x_2| - 2 + 2^{2-n}\right), \qquad \mathcal{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the gradients are given by

$$\nabla\ell^n(x_1, x_2) = \nabla\ell^1(2^{n-1}x_1, 2^{n-1}x_2) = \nabla\psi\left(2^{n-1}x_1 \bmod 1 - \frac{1}{2}, 2^{n-1}x_2\right)$$

and

$$\begin{aligned} \nabla\varphi^n(x_1, x_2) &= \mathcal{T}^{n-1}\nabla\ell^{n-1}\left(x_1, |x_2| - 2 + 2^{2-n}\right)\begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(x_2) \end{pmatrix} \\ &= \mathcal{T}^{n-1}\nabla\psi\left(2^{n-2}x_1 \bmod 1 - \frac{1}{2}, 2^{n-2}|x_2| - 2^{n-1} + 1\right)\begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(x_2) \end{pmatrix}. \end{aligned}$$

Hence,

$$\det \nabla\varphi^1(x_1,x_2) = \begin{cases} +f_1'(x_1)f_2'(x_2), & |x|_1 \leqslant 1, \\ -f_2'(x_1)f_1'(x_2), & |x|_1 \geqslant 1, \end{cases}$$

$$\det \nabla\varphi^n(x_1,x_2) = (-1)^{n-1}\operatorname{sgn}(x_2)\det\nabla\psi\left(2^{n-2}x_1 \mod 1 - \frac{1}{2}, 2^{n-2}|x_2| - 2^{n-1}+1\right)$$

$$= \begin{cases} +(-1)^{n-1}\operatorname{sgn}(x_2)f_3'(y_1)f_2'(y_2) & |y|_1 \geqslant \frac{1}{2}, \\ -(-1)^{n-1}\operatorname{sgn}(x_2)f_2'(y_1)f_3'(y_2) & |y|_1 \leqslant \frac{1}{2}, \end{cases}$$

where

$$y(x) = \left(2^{n-2}x_1 \mod 1 - \frac{1}{2}, 2^{n-2}|x_2| - 2^{n-1}+1\right)^{\mathsf{T}}.$$

This allows us to calculate the Jacobian on $\mathcal{L}$. To get the Jacobian on the rest of Q, we use

$$\nabla(\varphi^*(|X_i|(x),|X_s|(x))) = \begin{cases} \nabla\varphi^*(|X_i|(x),|X_s|(x))\cdot\begin{pmatrix}\operatorname{sgn}(x_1) & 0 \\ 0 & \operatorname{sgn}(x_2)\end{pmatrix}, & |x_1| \leqslant |x_2|, \\ \nabla\varphi^*(|X_i|(x),|X_s|(x))\cdot\begin{pmatrix}0 & \operatorname{sgn}(x_2) \\ \operatorname{sgn}(x_1) & 0\end{pmatrix}, & |x_1| \geqslant |x_2|, \end{cases}$$

so

$$\det\nabla\varphi^*(x) = \begin{cases} +\det\nabla\varphi^n(|x_1|,|x_2|)\cdot\operatorname{sgn}(x_1)\operatorname{sgn}(x_2), & x\in\mathcal{L}^n, \\ -\det\nabla\varphi^n(|x_2|,|x_1|)\cdot\operatorname{sgn}(x_1)\operatorname{sgn}(x_2), & x\in\mathcal{A}^n-\mathcal{L}^n. \end{cases}$$

Now that we have constructed one solution φ^* to (4.4), we can also derive an infinite family of solutions to (4.4).

Proposition 4.15. *Let $\varphi_{inn}: Q\to Q$ satisfy*

$$\begin{cases} \nabla\varphi_{inn}(x)\in \mathrm{O}(2), & \text{a.e. } x\in Q, \\ \varphi_{inn}(x)\in\partial Q, & x\in\partial Q, \end{cases}$$

and $\varphi_{out}: Q\to Q$ satisfy

$$\begin{cases} \nabla\varphi_{out}(x)\in \mathrm{O}(2), & \text{a.e. } x\in Q, \\ \varphi_{out}(0) = 0. \end{cases}$$

Then $\varphi = \varphi_{out}\circ\varphi^*\circ\varphi_{inn}$ solves (4.4).

Proof. By the group properties of $\mathrm{O}(2)$, we have

$$\nabla\varphi(x) = \nabla\varphi_{\mathrm{out}}(\varphi^*(\varphi_{\mathrm{inn}}(x)))\nabla\varphi^*(\varphi_{\mathrm{inn}}(x))\nabla\varphi_{\mathrm{inn}}(x)\in\mathrm{O}(2),$$

for almost every $x\in Q$. If $x\in\partial Q$, then $\varphi_{\mathrm{inn}}(x)\in\partial Q$ and so

$$\varphi(x) = \varphi_{\mathrm{out}}(\varphi^*(\varphi_{\mathrm{inn}}(x))) = \varphi_{\mathrm{out}}(0) = 0.$$

□

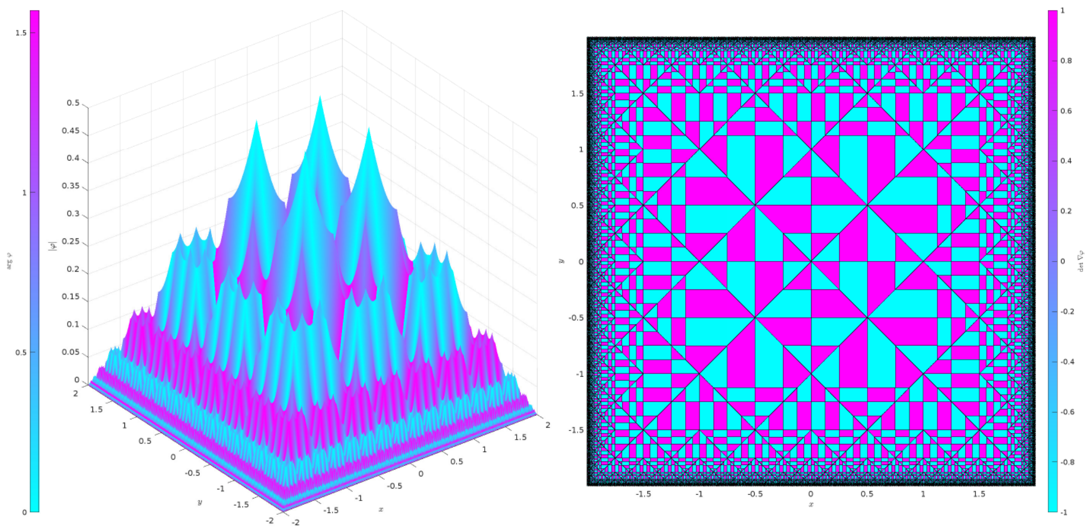

Figure 4.1: Plot of the solution φ^* to (4.4) and its Jacobian $\det \nabla\varphi^*$.

4.3. Numerical Results

Due to the arbitrarily frequent transitions between Q^+ and Q^- near the boundary, many numerical integration techniques will struggle to evaluate the integrals needed to apply Corollary 4.5. Recall that the necessary condition takes the form

$$\mathbb{E}_{\lambda \mathrm{p}} \geqslant 0 \quad \Rightarrow \quad |\lambda| \leqslant \frac{2m}{\kappa}, \qquad \kappa := \left| \fint_{Q^+} \mathrm{P}(y)\,\mathrm{d}y - \fint_{Q^-} \mathrm{P}(y)\,\mathrm{d}y \right|.$$

One solution to this problem is to split Q into a core square $Q_0 = \bigcup_{k=1}^{n} \mathcal{A}_k$ and a perimeter (square) annulus $\tilde{Q} = Q \backslash Q_0$, where we pick $n \in \mathbb{N}$ sufficiently large. On $\tilde{Q}$, we bound the integral term (4.5) required for the necessary condition using

$$\left| \fint_{Q^+} \mathrm{P}|_{\tilde{Q}}(y)\,\mathrm{d}y - \fint_{Q^-} \mathrm{P}|_{\tilde{Q}}(y)\,\mathrm{d}y \right| \leqslant \frac{2}{|Q|} \int_{\tilde{Q}} |\mathrm{P}(y)|\,\mathrm{d}y \leqslant 2\frac{|\tilde{Q}|}{|Q|} \cdot \left\| \mathrm{P}|_{\tilde{Q}} \right\|_\infty \leqslant 2m \frac{|\tilde{Q}|}{|Q|} \cdot \|\mathrm{p}\|_\infty ,$$

assuming that the pressure function is bounded. Then on Q_0, we only have a finite number of regions to integrate over, which can be handled by a computer algebra package. This will give an error bound of

$$\epsilon = 2m \left(2^{1-n} - 2^{-2n}\right) \|\mathrm{p}\|_\infty , \tag{4.7}$$

from the true value of κ in Corollary 4.5. We can the consider the worst-case scenario, given by

$$\kappa = \max\{\kappa_0 - \epsilon, 0\},$$

where κ_0 is the value for κ on Q_0. If we then have $\kappa > 0$, we can apply Corollary 4.5 to get a valid necessary condition of the parameter λ.

We shall now explore one specific example, namely, the one-parameter family of pressure functions $\mathsf{p}_\lambda = \lambda \mathsf{p}$ with

$$\mathsf{p}(x) = |x| \qquad x \in B.$$

For demonstration purposes, we will take $n = 64$ and $m = 2$ squares given by

$$Q_1 = [-0.5, +0.5]^2 + (0.1, 0.3)^\mathsf{T}, \qquad Q_2 = [-0.3, +0.3]^2 - (0.3, 0.5)^\mathsf{T}.$$

Using the error bound given in (4.7), we approximate κ as

$$\kappa \approx 2.71951 \times 10^{-5} \pm 4.33681 \times 10^{-19},$$

and so we get the necessary condition

$$\mathbb{E}_{\mathsf{p}_\lambda} \geqslant 0 \quad \Rightarrow \quad \lambda \leqslant 147075.$$

5 Conclusions and Outlook

Recall that our primary goal was to derive examples of mean Hadamard inequalities, that is, expressions of the form

$$\mathbb{E}_{\mathsf{p}}(\varphi) := \int_B \frac{1}{2}|\nabla\varphi|^2 + \mathsf{p}\det\nabla\varphi\,\mathrm{d}x \geqslant 0 \qquad \forall\varphi\in H_0^1(B;\mathbb{R}^2),$$

for some pressure function $\mathsf{p} \in \mathrm{BMO}(B)$. In Chapter 2, we derived several general families of pressure functions, taking the forms

$$\{\mathsf{p}\in W^{1,1}(B) : \|w\nabla\mathsf{p}\|_\infty \leqslant C\},$$

and

$$\{\mathsf{p}\in W^{1,1}(B) : w(r)\mathsf{p}(r) \leqslant \rho(r) \text{ and } \mathsf{p}(x) = \mathsf{p}(|x|)\},$$

that give rise to such inequalities. Particular care was taken to find explicit bounds for the right-hand sides C and ρ, for a given weight w. Furthermore, necessary conditions on w were derived to ensure the choice of weight would be valid. We also considered some more specific examples of forms the pressure could take, such as a monomial or a logarithmic function of $|x|$ and derived sufficient conditions on the parameters of these forms of pressure to obtain a mean Hadamard inequality. Some time was also taken to relate these results and techniques back to the problem of minimising the Dirichlet energy in a Jacobian constrained class.

In Chapter 3, we considered the much more general case of a pressure function $\mathsf{p}\in\mathrm{BMO}(B)$ that may not be differentiable. Here, we used techniques in Harmonic Analysis such as compensated compactness and BMO-$\mathcal{H}^1$ duality to find an explicit bound for a constant C, such that

$$\{\mathsf{p}\in\mathrm{BMO}(B) : [\mathsf{p}]_{\mathrm{BMO}} \leqslant C\}$$

is a valid family of pressure functions for a mean Hadamard inequality. The obtained bound for C was relatively large and could most likely be reduced by optimising some of the steps in the derivation, such as the use of a covering lemma for a Calderon-Zygmund type decomposition.

Finally, in Chapter 4, we explored the converse result, that is, pressure functions for which we do not obtain a mean Hadamard inequality. We did this by focussing on a specific example of the input map φ, that satisfied the PDI

$$\begin{cases} \nabla\varphi(x) \in \mathrm{O}(2), & \text{a.e. } x \in B, \\ \varphi(x) = 0, & x \in \partial B. \end{cases}$$

We constructed an explicit example of a solution to this PDI on a square domain, making use of previously established techniques, and then stitched multiple copies of this solution together to obtain a solution on the ball B. By approximating the integral in the excess functional, we were able to derive explicit examples of pressure functions for which the corresponding mean Hadamard inequality does not hold. The method used here could be further optimised by constructing additional solutions to the PDI and stitching them together in a more optimal way, which would depend on the form of the pressure function in question.

We have focussed specifically on the example of regularity and dimension both being two ($n = p = 2$) but mean Hadamard inequalities can exist in other dimensions. More generally, a mean Hadamard inequality, takes the form

$$\mathbb{E}_{\mathsf{p}}(\varphi) := \int_B \frac{1}{n} |\nabla\varphi|^n + \mathsf{p} \det \nabla\varphi \, \mathrm{d}x \geqslant 0 \qquad \forall \varphi \in W_0^{1,n}(B; \mathbb{R}^n).$$

The pre-factor of $\frac{1}{n}$ is just a convention inspired by the variational problem associated with the p-Laplacian and can be omitted by rescaling p. The important observation here is that we are still in the critical case of the Sobolev embedding ($n = p$) and we have homogeneity in the integrand as both terms are of degree n. Many of the results that we have derived can be generalised to higher dimensions but, naturally, there will also be some challenges to doing this.

- The Euler-Lagrange equation associated with $\mathbb{E}_{\mathsf{p}}$ becomes non-linear when $n > 2$. This introduces additional complexity when trying to use PDE theory to study this functional. In relation to this, we also have that the cofactor operator is not linear on $\mathbb{R}^{n \times n}$ for $n > 2$.

- When expanding the terms of $\mathbb{E}_{\mathsf{p}}$ using Fourier series, not only will the terms be more complicated due to the increase in dimension of the domain, but we will also have to deal with integrals of products of more than two trigonometric functions. This will require more generalised orthogonality relations to handle the calculations.

- The explicit bound for the scale factor in the covering lemma used would need to be generalised to higher dimensions. This should be achievable through a counting argument. The rest of results in Chapter 3 are independent of the domain geometry.

- The construction in Chapter 4 came from simplifying a result in the literature to work in two dimensions instead of three. This means that we could immediately generalise these results to three dimensions but going beyond this would require the development of a more generalised construction technique.

A further generalisation would be to consider completely different geometry for the domain, for example, an annulus or a torus. This should be possible as we can make use of conformal mappings. For a square domain, the equivalent results in Chapter 4 could be produced by simply skipping the stitching step and using the constructed solution directly.

It is worth noting that the results presented here also contribute to the progression of certain topics in elasticity theory. Through the derivation of mean Hadamard inequalities, we have constructed a wide range of explicit examples of Jacobian constrained variational problems that arise in elasticity, where Jacobian constraints describe incompressibility-like conditions for materials. Furthermore, we have constructively shown the existence of minimisers to a family of polyconvex functionals (the excess functional for various pressure functions), which are often arise as stored energy functionals. It may be that the techniques used here could be adapted to other polyconvex functionals, providing deeper insights into some of the open problems in the field of elasticity.

References

[1] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions: With formulas, graphs and mathematical tables [conference under the auspices of the National science foundation and the Massachussetts institute of technology]*, unabridged, unaltered and corr. republ. of the 1964 ed., Dover Books on Advanced Mathematics, Dover publ, New York, 1972.

[2] George E. Andrews, Richard Askey, and Ranjan Roy, *Special functions*, Encyclopedia of mathematics and its applications, no. v. 71, Cambridge University Press, Cambridge, UK, 1999.

[3] T V Anoop, Ujjal Das, and Subhajit Roy, *On Weighted Orlicz-Sobolev inequalities*, 2023, doi.

[4] Nakhlé H. Asmar, *Partial differential equations with Fourier series and boundary value problems*, 2 ed., Pearson Prentice-Hall, Upper Saddle River, NJ, 2005.

[5] Thierry Aubin and Yan Yan Li, *On the best Sobolev inequality*, Journal de Mathématiques Pures et Appliquées **78** (1999), no. 4, 353–387, doi.

[6] Andrew Baker, *Matrix groups: An introduction to Lie group theory*, Springer Undergraduate Mathematics Series, Springer, London, 2002.

[7] John M. Ball, *Some Open Problems in Elasticity*, Geometry, Mechanics, and Dynamics (Paul Newton, Philip Holmes, and Alan Weinstein, eds.), Springer-Verlag, New York, 2002, pp. 3–59 (en), doi.

[8] Yoav Benyamini and Joram Lindenstrauss, *Geometric nonlinear functional analysis*, American Mathematical Society Colloqium Publications, no. 48, American mathematical society, Providence (R.I.), 2000.

[9] Jonathan Bevan, *On double-covering stationary points of a constrained Dirichlet energy*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire **31** (2014), no. 2, 391–411, doi.

[10] Jonathan Bevan, Martin Kružík, and Jan Valdman, *Hadamard's inequality in the mean*, Nonlinear Analysis **243** (2024), 113523, doi.

[11] William E. Boyce and Richard C. DiPrima, *Elementary differential equations and boundary value problems*, 10 ed., Wiley, Hoboken, NJ, 2013.

[12] T. A. A. Broadbent, *Dilogarithms and Associated Functions. By L. Lewin. Pp. xvi, 353. 65s. 1958. (Macdonald)*, The Mathematical Gazette **43** (1959), no. 346, 312–313 (en), doi.

[13] James Ward Brown and Ruel Vance Churchill, *Complex variables and applications*, 6 ed., Churchill-Brown Series, McGraw Hill, New York, 1996.

[14] Arrigo Cellina and Stefania Perrotta, *On a problem of potential wells*, J. Convex Anal **2** (1995), no. 1-2, 103–115.

[15] Roger Ceschi and Jean-Luc Gautier, *Fourier analysis*, Digital Signal and Image Processing Series, Wiley, Hoboken, NJ, 2017, doi.

[16] Ronald R. Coifman, P. L. Lions, Yves Meyer, and Stephen Semmes, *Compensated compactness and Hardy spaces*, Journal de Mathématiques Pures et Appliquées **72** (1993), no. 3, 247–286.

[17] Bernard Dacorogna, *Direct methods in the calculus of variations*, Applied mathematical sciences, no. 78, Springer, Berlin Heidelberg, 1989 (eng).

[18] Bernard Dacorogna and Paolo Marcellini, *Implicit Partial Differential Equations*, Birkhäuser Boston, Boston, MA, 1999, doi.

[19] Bernard Dacorogna and Jürgen Moser, *On a partial differential equation involving the Jacobian determinant*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire **7** (1990), no. 1, 1–26, doi.

[20] R. E. Edwards, *Trigonometric Series and Fourier Series*, Fourier Series, vol. 64, Springer New York, New York, NY, 1979, Series Title: Graduate Texts in Mathematics, pp. 1–13, doi.

[21] Lawrence C. Evans, *Partial differential equations*, 2 ed., Graduate studies in mathematics, no. v. 19, American Mathematical Society, Providence, R.I, 2010.

[22] Daniel Faraco and Xiao Zhong, *Geometric rigidity of conformal matrices*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 5, 4** (2005), no. 4, 557–585 (en).

[23] David J. H. Garling, *Inequalities: A journey into linear analysis*, Cambridge University Press, Cambridge New York, 2007.

[24] S. Yu. Graf and I. A. Nikitin, *Planar Harmonic Mappings With a Given Jacobian*, Issues of Analysis **30** (2023), no. 3, 69–85, doi.

[25] G. H. Hardy, *Weierstrass's Non-Differentiable Function*, Transactions of the American Mathematical Society **17** (1916), no. 3, 301, doi.

[26] Michiel Hazewinkel (ed.), *Encyclopaedia of mathematics*, unabridged reprint of the original 10-vol. hardbound library ed., Kluwer, Dordrecht, 1995.

[27] Don Hong, Jianzhong Wang, and Robert B. Gardner, *Real analysis: With an introduction to wavelets and applications*, Elsevier Academic Press, Amsterdam Boston, 2005.

[28] Roger A. Horn and Charles R. Johnson, *Topics in matrix analysis*, transferred to digital printing ed., Cambridge Univ. Press, Cambridge, 2010.

[29] Tobias Lamm, *Geometric Variational Problems*, 2007.

[30] Haoxiang Lin, *A simple proof on the inequality of arithmetic and geometric means*, 2012, doi.

[31] James M. Ortega, *Matrix theory: A second course*, The University Series in Mathematics, Plenum Press, New York London, 1991.

[32] Tohru Ozawa and Taiki Takeuchi, *A new proof of the Gagliardo–Nirenberg and Sobolev inequalities: Heat semigroup approach*, Proceedings of the American Mathematical Society, Series B **11** (2024), no. 33, 371–377, doi.

[33] Lars-Erik Persson and Natasha Samko, *ON HARDY-TYPE INEQUALITIES AS AN INTELLECTUAL ADVENTURE FOR 100 YEARS*, Journal of Mathematical Sciences **280** (2024), no. 2, 180–197, doi.

[34] Filip Rindler, *Calculus of Variations*, Universitext, Springer International Publishing, Cham, 2018, doi.

[35] Michael Ruzhansky and Durvudkhan Suragan, *Critical Hardy inequalities*, 2016, doi.

[36] Harry M. Schey and Harry Moritz Schey, *Div, grad, curl, and all that: An informal text on vector calculus*, 3 ed., Norton, New York London, 1997.

[37] J. Michael Steele, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, 1 ed., Cambridge University Press, April 2004, doi.

[38] Elias M. Stein and Timothy S. Murphy, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, [Princeton Mathematical Series, no. 43], Princeton University Press, Princeton, N.J, 1993.

[39] Gilbert Strang, *Introduction to linear algebra*, 6th ed ed., Wellesley-Cambridge press, Wellesley, Mass, 2023.

[40] Vladimír Šverák, *On the Problem of Two Wells*, Microstructure and Phase Transition (Avner Friedman, Willard Miller, David Kinderlehrer, Richard James, Mitchell Luskin, and Jerry L. Ericksen, eds.), vol. 54, Springer New York, New York, NY, 1993, pp. 183–189, doi.

[41] Eberhard Zeidler, *Nonlinear Functional Analysis and its Applications*, Springer New York, New York, NY, 1985, doi.